Overview of Multivariable Calculus

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The following is a brief overview of the contents of a typical course in Multivariable Calculus that I hope will help students organize their studies and understand what the course is about; it is a cleaned up version of a study guide that I've been working on over several semesters. It is targeted at being accessible to first- or second-year undergraduates. If you have any comments, or catch any mistakes, please let me know! I have often used James Stewart's line of calculus books in teaching and so some of the organization of these notes are parallel to those, but I have made significant changes and the content is all original. I'd highly recommend the free, online Paul's Online Math Notes from Lamar University for students looking for further help.

1 Introduction to higher-dimensional geometry

Multivariable Calculus courses will often start with an introductory section to vector geometry; this material can be presented much earlier on (I've seen it taught as part of a junior high algebra course!) but many students have not seen it before. My notes for this section are a bit sparse.

1.1 Intro to Vector Geometry

- 3D Coordinates
  - Definition of points in $\mathbb{R}^3$ and $\mathbb{R}^2$
  - Coordinate planes (e.g. xy-plane)
  - Equation of a sphere $x^2 + y^2 + z^2 = R^2$, or $((x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2)$
  - Right-hand rule for drawing coordinate system
- Vectors
  - Definition of a vector
  - Coordinate form: $\vec{v} = \langle v_1, v_2, v_3\rangle$ in $\mathbb{R}^3$ or $\vec{v} = \langle v_1, v_2\rangle$ in $\mathbb{R}^2$
  - Vector has length and direction (NOT position!)
  - The magnitude (or length) of a vector is given by

$$\|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2} \qquad (\text{in } \mathbb{R}^3)$$

or

$$\|v\| = \sqrt{v_1^2 + v_2^2} \qquad (\text{in } \mathbb{R}^2)$$

  - Vector arithmetic (addition, subtraction, scalar addition)

- Dot Product
  - Coordinate formula $\vec{v} \cdot \vec{u} = v_1u_1 + v_2u_2 + v_3u_3$
  - Angle formula $\vec{v} \cdot \vec{u} = \|\vec{v}\|\|\vec{u}\| \cos(\theta)$
  - Dot product is a scalar (NOT a vector!)
  - The main purpose of the dot product is to determine angle between vectors
  - If the dot product of two vectors is zero, they are orthogonal

- Cross Product
  - Coordinate formula

$$\vec{v} \times \vec{u} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{pmatrix} = \langle v_2u_3 - v_3u_2, v_3u_1 - v_1u_3, v_1u_2 - v_2u_1 \rangle$$

  - Length formula $\|\vec{v} \times \vec{u}\| = \|\vec{v}\|\|\vec{u}\| \sin(\theta)$
  - Geometric properties: $\vec{v} \times \vec{u}$ is orthogonal to both $\vec{v}$ and $\vec{u}$, pointing in the direction determined by the right-hand rule
  - The main purpose of the cross product is to make a new vector orthogonal to the originals (useful especially for finding normal vector to a plane)
  - Warning: The cross product is NOT commutative! In fact, it is anticommutative, that is

$$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$$

1.2 Curves

- Vector-Valued Functions
  - Vector-valued functions are functions $\mathbb{R} \to \mathbb{R}^2$ or $\mathbb{R} \to \mathbb{R}^3$
  - In component form, we write

$$\vec{r}(t) = \langle f(t), g(t) \rangle$$

or

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

  - They describe curves in space, you can think of t as time, and the output as the position of a particle moving along the curve

- Limits can be taken componentwise, that is

$$\lim_{t\to a} \vec{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \right\rangle$$

- Know some examples, especially straight lines, circles, and helices.

• Derivatives and Integrals of Vector-Valued Functions

- We can take derivatives and integrals componentwise, that is

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

and

$$\int \vec{r}(t)\ dt = \left\langle \int f(t)\ dt, \int g(t)\ dt, \int h(t)\ dt \right\rangle$$

- The derivative of a vector-valued function gives you a "velocity" vector; it points in the direction the curve is moving, and its magnitude is the speed of a particle moving along the curve
- The product rule works for dot and cross products! Be careful to get the order correct for the cross product, since it is anticommutative.

$$\frac{d}{dt}(\vec{v}(t) \cdot \vec{u}(t)) = \vec{v}\,'(t) \cdot \vec{u}(t) + \vec{v}(t) \cdot \vec{u}\,'(t)$$

and

$$\frac{d}{dt}(\vec{v}(t) \times \vec{u}(t)) = \vec{v}\,'(t) \times \vec{u}(t) + \vec{v}(t) \times \vec{u}\,'(t)$$

- The integral of a vector-valued function is best understood through the fundamental theorem of calculus. If you take the integral of the derivative of a vector valued function, you get displacement. That is,

$$\int_a^b \vec{r}\,'(t)\ dt = \vec{r}(b) - \vec{r}(a)$$

• Arc Length

- The distance traveled by a particle moving along the curve $\vec{r}(t)$ from time $t = a$ to $t = b$ is given by the <u>arc length formula</u>

$$L = \int_a^b \|\vec{r}\,'(t)\|\ dt$$

- Make sure you can apply this formula by hand and with a calculator; review integration if you're having a hard time!
- This often comes up in story problems e.g. "how far did the ball roll down the hill if the side of the hill is shaped like the curve $\vec{r}(t)$?"

• Motion in Space

- Velocity and acceleration can be expressed as vectors
- Velocity is the derivative of position, and acceleration is the derivative of velocity

$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$$

- In the other direction, velocity is the integral of acceleration (you need an initial value for velocity) and position is the integral of velocity (you need an initial value for position.)

$$\vec{r}(t) = \int \vec{v}(t) = \iint \vec{a}(t)$$

- Speed is just the magnitude of velocity. Make sure to remember, velocity is a vector but speed is a scalar!

1.3 Surfaces

- Equations of Lines and Planes
  - EQ of a line through point $A = (a_1, a_2, a_3)$ and parallel to vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$:

    $$\vec{r}(t) = \vec{A} + t\vec{v}$$

    or

    $$\begin{cases} x(t) = a_1 + tv_1 \\ y(t) = a_2 + tv_2 \\ z(t) = a_2 + tv_2 \end{cases}$$

  - EQ of a plane through points $P = (x, y, z)$ and $A = (x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c \rangle$:

    $$\vec{n} \cdot \overrightarrow{AP} = 0$$

    or

    $$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

  - Know how to find EQ of a line from 2 points
  - Know how to find EQ of a plane from 3 points
- Functions and Surfaces
  - Multivariate functions of two variables $f(x, y)$ can be thought of as surfaces $z = f(x, y)$. We often think of them as their graphs $(x, y, f(x, y))$.
  - We can also define surfaces implicitly using functions of three variables, e.g. $x^2 + y^2 + z^2 = 1$ is a sphere of radius 1 centered at the origin

- Functions of three variables that are polynomials of order 2 are called "quadric surfaces", you should know the following examples:

$$\begin{cases} d = ax + by + cz & \text{planes} \\ 1 = ax^2 + by^2 + cz^2 & \text{ellipses} \\ z = ax^2 + by^2 & \text{elliptic paraboloids} \\ z^2 = ax^2 + by^2 & \text{cones} \end{cases}$$

  (planes aren't really quadric surfaces, but you should know them)
- Notice, some quadric surfaces can't be written as graphs of functions. For example, if you take the cone $z^2 = x^2 + y^2$ and try to solve for z, you get

$$z = \pm\sqrt{x^2 + y^2}$$

  which isn't a function. Make sure you know what the graph of

$$z = +\sqrt{x^2 + y^2}$$

  looks like.

- Polar, Cyclindrical, and Spherical Coordinates
  - To express a point in $\mathbb{R}^2$ (or $\mathbb{R}^3$) you need 2 (or 3) numbers; usually we use Cartesian coordinates (x, y) (or (x, y, z).
  - In $\mathbb{R}^2$ we can use polar coordinates (r, θ) instead, where r represents radius and θ represents counterclockwise angle measured from the positive x-axis.
  - This can be really useful for rotationally symmetric objects, like circles and ellipses! There are also downsides, such as the fact that points aren't unique (for example, $(2, 0)$ and $(2, 2\pi)$ both represent the same point.)
  - In $\mathbb{R}^3$ we have two new coordinate systems, cylindrical (r, θ, z) and spherical (ρ, θ, ϕ).
  - To imagine cylindrical coordinates (r, θ, z), first imagine polar coordinates in the xy-plane, then just move in a straight line in the z-direction. The most useful coordinate transformations are

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \\ r^2 &= x^2 + y^2 \\ \theta &= \tan^{-1}(y/x) \end{aligned}$$

  - Spherical coordinates (ρ, θ, ϕ) can be a little trickier; ρ is the straight-line distance from the origin, and θ, ϕ are like longitude and latitude on a sphere. To be specific, θ is the same as in cylindrical coordinates and ϕ is the angle from the positive z-axis. The most useful coordinate transformations are

$$\begin{aligned} x &= \rho\cos(\theta)\sin(\phi) \\ y &= \rho\sin(\theta)\sin(\phi) \\ z &= \rho\cos(\phi) \\ \rho^2 &= x^2 + y^2 + z^2 \\ r &= \rho\sin(\phi) \\ \phi &= \tan^{-1}\left(\frac{r}{z}\right) = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) \end{aligned}$$

  - Make sure you can take surfaces in one coordinate system and write them in another! This comes down to finding the right coordinate substitutions (and simplifying!)

- Parametric Surfaces
  - A parametric surface is a map $\mathbb{R}^2 \to \mathbb{R}^3$
  - We usually write this $\vec{r}(u,v) = \langle f(u,v), g(u,v), h(u,v)\rangle$
  - Since we have two variables, we can "move in two directions", which is why we get a surface (instead of, say, a curve)
  - One "easy" way to write a surface parametrically is to find an equation $z = f(x,y)$ for it, and then set $x = u, y = v, z = f(u,v)$; note that this only works if you can solve for z.
  - Another option is to choose $u = r, v = \theta$ in cylindrical coordinates and find expressions for x, y, z. For example, since a cone $z^2 = x^2+y^2$ can be written in cylindrical coordinates as $z = r$, you can write a cone parametrically as

    $$\vec{r}(u,v) = \langle u\cos(v), u\sin(v), u\rangle$$

    using the standard coordinate transformations $x = r\cos(\theta), y = r\sin(\theta)$ for x and y.

2 Partial Derivatives

Now we get into the "meat" of calculus, derivatives! They often work just like they did in Calculus 1 and 2, but important details are different, and we can use them for a lot of more interesting things.

2.1 Theory of Partial Differentiation

- Multivariate Functions
  - A multivariate function is a map $\mathbb{R}^2 \to \mathbb{R}$ (or $\mathbb{R}^3 \to \mathbb{R}$,) that is it takes 2 (or 3) real numbers as variables and returns 1 real number as an output.
  - You want to be able to interpret these:
    * Verbally: a story problem, stated as a sentence (e.g. the wind speed is a function of position and time.)
    * Numerically: a table of values, often expressed in a chart.
    * Algebraically: written as an explicit formula (e.g. $f(x, y) = x^2y$.)
    * Visually: a graph, a surface, level curves, etc.
  - Level curves are curves determined by setting a multivariate function to a constant value. The example to think of is topographical maps. Make sure you can interpret properties of a function given a picture of its level curves, or sketch a graph of a function's level curves yourself.
- Limits, Continuity
  - Limits have the same technical definition they did in earlier classes.
  - Unfortunately, the idea of a left limit and a right limit no longer make sense, in fact we have to take limits from all possible directions.
  - In practice, this isn't possible, so we either have to simplify the expression algebraically, or try to take a limit in polar (or spherical) coordinates, since as $r \to 0$ (or $\rho \to 0$) we get all directions, just make sure that your answer doesn't depend on θ (or (θ, ϕ)).
  - A function is continuous at a point if the function at that point equals its limit at the point.
- Partial Derivatives
  - We can take a partial derivative of a multivariate function by holding all but one variable constant and taking the normal derivative.
  - This is the basic building block of differentiation and the rest of the class, make sure you can do this!
  - There are a lot of different notations for partial derivatives, don't let them confuse you. The most common are the italic $\frac{\partial}{\partial x} f$ and the subscript f_x.
  - Partial derivatives can be interpreted in different ways, the easiest visual is to think of $z = f(x, y)$ as a surface, then f_x is the slope of the surface in the x-direction. Notice that this doesn't really work for functions of 3 or more variables, since they can't be thought of as surfaces.

- We also have the very important Clairaut's Theorem, which says that as long as $\frac{\partial^2 f}{\partial x \partial y} = f_{xy}$ and $\frac{\partial^2 f}{\partial y \partial x} = f_{yx}$ are both continuous functions, then they are equal. In other words, you can take the derivatives in either order and still get the same result!

- Chain Rule
  - This is a generalization of the chain rule from earlier classes that lets you take derivatives of compositions of functions with multiple variables.
  - The easiest approach is to draw the tree, then find all paths that terminate in the variable that you're interested in.
  - Alternatively, we have this general formula for $f(x, y)$ where $x = x(u, v), y = y(u, v)$:

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

  - Similar formulas hold if you have more variables. Keep in mind that if you imagine "cancelling" each term, you should get the "correct" derivative, e.g. $\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}$ "cancels" to $\frac{\partial f}{\partial u}$ (just remember that you can't actually cancel, because these aren't actually fractions, this is just a useful check for mistakes.)

2.2 Applications of Partial Derivatives

- Tangent Planes, Linearization
  - Tangent planes are just like tangent lines in earlier classes; tangent lines touch a curve at a single point (locally) while tangent planes touch a surface at a single point (locally.)
  - For the tangent plane to the surface $z = f(x, y)$ at the point $(x_0, y_0, f(x_0, y_0))$ we have the equation

$$z - f(x_0, y_0) = \left(\frac{\partial f}{\partial x}(x_0, y_0)\right)(x - x_0) + \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)(y - y_0)$$

  - They serve as the best linear approximation to the surface; that is near the point they are tangent to they are a good approximation of the function. Make sure that you can use them to approximate functions! (E.g. approximate the value of $1.1\sqrt{3.8}$ using the linearization of the function $f(x, y) = x\sqrt{y}$ at the point $(1, 4)$.)

- Directional Derivatives and Gradient Vectors
  - Partial derivatives allow us to determine the rate of change (slope) of a function along the x- and y-directions, but what about in other directions?

- In order to find the rate of change of a function $f(x,y)$ in the direction $\vec{v} = \langle v_1, v_2 \rangle$ we use the directional derivative

$$D_{\vec{v}} f = v_1 \frac{\partial f}{\partial x} + v_2 \frac{\partial f}{\partial y}$$

  if $\vec{v}$ is a unit vector.
- This formula comes from adapting the limit definition for partial derivatives.
- To make our lives easier, we define the gradient vector of a function f as

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

- Using the gradient, we can redefine the directional derivative (for any vector $\vec{v}$, not just unit vectors)

$$D_{\vec{v}} f = \frac{\nabla f \cdot \vec{v}}{\|\vec{v}\|}$$

- The gradient has other uses; the **direction** of greatest rate of change of a function f will always be in the direction of the gradient ∇f, and the **greatest rate of change** will be its magnitude $\|\nabla f\|$.
- Also, the gradient of a function ∇f will always be orthogonal to its level curves (this really helps when interpreting graphs of level functions!!)
- The gradient vector also makes sense with more variables, if $f(x,y,z)$ is a function of 3 variables we define

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

  and so on.

- Extrema of Multivariate Functions
  - One of the major goals of elementary calculus is to find extrema of functions; that is determine the maximum and minimum values, and where they occur.
  - In Calculus 1, we had the 1st and 2nd derivative tests; these relied on first finding the critical points (where the derivative was 0) and then checking whether these were minima, maxima, or neither.
  - The first step still works; we call a point (a,b) a critical point of f if $\nabla f = \vec{0}$ which is the same as saying each of first partial derivatives are 0:

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$$

- The 1st derivative test from Calculus 1 won't really work, since we'd have to test the value of the derivative from every direction! In Calculus 1 this was fine, since the only directions were left and right, but now we have an entire circle of directions.
- The 2nd derivative test will work, with a bit of modification. The first step will be to check the value of

$$D = f_{xx} f_{yy} - (f_{xy})^2$$

at each of the critical points (a, b). We have three cases:

  * If $D(a, b) > 0$ then (a, b) is either a maximum or a minimum. To check, use the usual 2nd derivative test with f_{xx} (or f_{yy} if you want, the answer will be the same (can you tell why?)), that is if $f_{xx}(a, b) > 0$ then (a, b) is a minimum, if $f(a, b) < 0$ then $f(a, b) < 0$ then (a, b) is a maximum. Take a second and see if you can figure out why f_{xx} won't ever be 0 here.
  * If $D(a, b) < 0$ then (a, b) is a saddle point. The example to think of is the origin on the hyperbolic paraboloid $z = x^2 - y^2$ (it looks like a horse saddle.)
  * If $D(a, b) = 0$ then we can't say anything about (a, b), and the test is "inconclusive;" we won't deal with this situation in this class.
- Sometimes you'll want to find all of the extrema under some condition, (e.g. $x^2 + y^2 < 4$, the interior of a disk with radius 2.) In this case, if you find a critical point that doesn't satisfy the condition, just skip it.

• Lagrange Multipliers

- Lagrange multipliers are a different technique for determining extreme values of a function. They only work if you have an extra condition with an equality (e.g. $x^2 + y^2 = 4$, the boundary of a disk with radius 2.) These are often used for finding extreme values at the boundary of a region.
- Given a function $f(x, y)$ and condition $g(x, y) = k$ (where k is a constant) you want to solve the system of equations

$$\begin{aligned} \frac{\partial f}{\partial x} &= \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} &= \lambda \frac{\partial g}{\partial y} \\ g(x, y) &= k \end{aligned}$$

Notice that this is a system with three equations, and three variables x, y, λ, so in principle it should be solvable. It's often not linear, so it can be tricky to solve. One big hint is to look for a term that appears in more than one equation, solve each for it, and then set the resulting equations equal.

- The value of λ isn't important for our purposes, but in some applications it is very useful.
- Once you've found all (x, y) that solve the system, check the value of f at each one. The largest and smallest are your maximum and minimum values of f on the boundary, respectively.
- One type of problem that you will often encounter is to find the extrema of a function f on the region $g(x, y) \leq k$. In this situation, use the 2nd derivative test to find the extrema on $g(x, y) < k$, and then use Lagrange multipliers to find the extrema on $g(x, y) = k$.

3 Multiple Integration

Integration takes more work than it did before, and a lot of visualization in order to properly write the bounds on higher-dimensional regions. This section may be the most useful to people going into work in STEM fields, I'd highly recommend you learn how to properly work with these tools.

3.1 Theory of Multiple Integration

- Definition of Double Integrals
  - The idea of a double integral is similar to that of a single integral: take a region of the xy-plane, break it up into small rectangles, and evaluate the value of a function $f(x, y)$ on each rectangle, then add them all up. Taking a limit as the number of rectangles goes to infinity gives us the integral:

$$\int\int_R f(x,y)\ dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\Delta x \Delta y$$

  - We can take a Midpoint approximation by picking some finite number of rectangles and adding the values across them. Make sure you can do this!
  - We can interpret a double integral as the volume of a solid between the surface $z = f(x, y)$ and the xy-plane, so if you can find the volume using geometric techniques you can use it to compute the integral.
- Iterated Integrals
  - Midpoint approximations are approximations, geometric techniques don't always work, and limits can be difficult (they have their *limitations*), so we want an easier way to compute double integrals.

- Fubini's theorem is the integral version of Clairaut's theorem, it says that as long as the functions are continuous and all of the integrals involved are finite, then over the rectangle $R = \{(x, y) \in \mathbb{R}^2, a \leq x \leq b, c \leq y \leq d\}$ the following statement holds:

$$\int\int_R f(x, y)\, dA = \int_a^b \left(\int_c^d f(x, y)\ dy \right) dx = \int_c^d \left(\int_a^b f(x, y)\ dx \right) dy$$

- This just means that we can perform one regular integral (holding one of the variables constant) and then the other, instead of trying to do both at once.
- Also, the order shouldn't matter!
- This is **the** main tool used in evaluating double (and higher order) integrals.

- Double Integration over General Regions
  - If your region is not rectangular, you can't just iterate two standard integrals; instead we try to write bounds for the region where one variable will be determined by the other.
  - For example: the triangle T with vertices $(0, 0), (1, 0), (0, 1)$ can be written $T = \{0 \leq x \leq 1, 0 \leq y \leq 1 - x\}$ or also as $T = \{0 \leq x \leq 1 - y, 0 \leq y \leq 1\}$ since the hypotenuse of the triangle is on the line $y = 1 - x$.
  - Another example: a circle C with radius 1 and centered at the origin can be written as $C = \{-1 \leq x \leq 1, -\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}\}$ or also as $C = \{-1 \leq y \leq 1, -\sqrt{1 - y^2} \leq x \leq \sqrt{1 - y^2}\}$
  - In general, one of your variables will always be bounded by two numbers, the other should include the first as a variable (unless the region is a rectangle.)
  - For example, over the circle C from before

$$\iint_C f(x, y)\ dA = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x, y)\ dydx$$

  - To visualize, you can think of $dydx$ as integrating along lines running vertically, and $dxdy$ as integrating along lines running horizontally; this can help you determine the bounds.
  - There is one particular problem where changing the order of integration is very useful:

$$\int f(x^2)\ dx$$

often can't be solved when f is an elementary function like e^x or $\sin(x)$, but the double integral

$$\int_0^a \int_y^a f(x^2)\ dxdy$$

can be solved by switching the order of integration so that

$$\int_0^a \int_0^x f(x^2)\ dydx = \int_0^a xf(x^2)\ dx = \frac{1}{2}\int_0^{a^2} f(u)\ du$$

where we use $u = x^2$. You have to be careful to change the order of integration properly, which almost always requires drawing the domain of integration. Don't be afraid to draw pictures!

- Double Integration in Polar Coordinates
  - Sometimes it is easier to describe a region in polar coordinates; this generally happens with figures that are symmetric around the origin.
  - For example, the circle C centered at the origin with radius 1 can be written as $C = \{0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$ or also as $C = \{-1 \leq r \leq 1, 0 \leq \theta \leq \pi\}$ (although this second version is often harder to use.)
  - To integrate, be careful to remember that $dA = r\ drd\theta$. This extra factor of r is a consequence of the fact that small regions $\Delta r \Delta\theta$ are not rectangles (like $\Delta x \Delta y$,) but sectors of annuli.
  - So, for example, integrating over C in polar coordinates becomes

$$\iint_C f(x,y)\ dA = \int_0^1 \int_0^{2\pi} f(r\cos\theta, r\sin\theta)\ r\ d\theta dr$$

  - Keep in mind, you'll have to rewrite your function in polar coordinates too!
- Triple Integration
  - Triple integrals are also defined using limits of Riemann sums (over 3-dimensional boxes.)
  - Fubini's theorem can still be applied, so the method of evaluating triple integrals is almost always using iterated integration.
  - Be careful when parameterizing the regions! I'd recommend looking over examples of this process, to make sure that you're getting the parameterization correct. This is the hardest part of triple integration.
- Triple Integration in Cylindrical and Spherical Coordinates
  - Just like using polar coordinates can be helpful for double integrals, cylindrical and spherical coordinates can be helpful for triple.

- Generally, cylindrical coordinates are useful when you have a region that is symmetric around an axis (usually the z-axis, but if your region is around a different axis you can just switch x, y, z.) The volume form is $dV = r\ drd\theta dz$.
- Similarly, spherical coordinates are useful when the region is symmetric around the origin (like a sphere.) The volume form is $dV = \rho^2 \sin\phi\ d\rho d\theta d\phi$.

- Change of Variables for Integration
  - This chapter covers how to change coordinate systems in general, and how to find the area form dA or volume form dV for new coordinate systems.
  - If you have a new coordinate system (a, b, c) and you want to change from (x, y, z), you determine functions $x(a, b, c), y(a, b, c), z(a, b, c)$ and then compute the following (called the determinant of the Jacobi matrix):

$$dV = \det \begin{pmatrix} x_a & x_b & x_c \\ y_a & y_b & y_c \\ z_a & z_b & z_c \end{pmatrix}$$

  - See if you can prove the formulas for dV in cylindrical and spherical coordinates!

3.2 Applications of Double Integration

- Center of Mass
- Moments of Inertia
- Probability distributions
- Surface Area
  - We can use double integrals to compute the surface area of a parametric surface $\vec{r}(u, v)$ using the formula

$$A = \iint_R \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\|\ dA$$

    where R describes bounds on (u, v) covering the entire surface.
  - This is a consequence of the fact that $\|\vec{u} \times \vec{v}\|$ is the area of the parallelogram formed by $\vec{u}$ and $\vec{v}$.
  - We'll be coming back to this idea soon in more generality.

4 Vector Calculus

This is where the real beauty of this class lies. We're able to generalize the Fundamental Theorem of Calculus, and the first ideas of differential geometry start to show through. If you are interested by this, you should consider taking higher level math courses!

4.1 Preliminaries and Line Integration

- Vector Fields
  - A vector field is an assignment of a vector to each point in space, or in other words a map $\mathbb{R}^2 \to \mathbb{R}^2$ or $\mathbb{R}^3 \to \mathbb{R}^3$.
  - You can visualize this as there being a 'field' of vectors, one at every point in space.
  - The usual notation for this is $\vec{v}(x, y) = \langle f(x, y), g(x, y)\rangle$ or $\vec{v}(x, y, z) = \langle f(x, y, z), g(x, y, z), h(x, y, z)\rangle$.
  - The most important example of this is the gradient! $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$ is a vector field.
  - A vector field $\vec{v}$ is called <u>conservative</u> if it is the gradient of some function f, that is if $\vec{v} = \nabla f$. If this is the case, we call f a <u>potential</u> for $\vec{v}$.
  - If $\vec{v} = \langle P(x, y), Q(x, y)\rangle$ and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ then $\vec{v}$ is conservative (this is because of Clairaut's Theorem!), you still have to use antidifferentiation to find the potential.
- Line Integrals
  - A line integral

    $$\int_C f(x, y)\ ds$$

    or

    $$\int_C \vec{v}(x, y) \cdot d\vec{s}$$

    expresses integration along a curve C.
  - They should probably be called "curve integrals" or "path integrals" since they work over any curve, but the standard name is "line integral."
  - In order to compute a line integral, you must:
    * Parameterize C as $\vec{r}(t)$ with bounds $a \leq t \leq b$
    * Write $d\vec{s} = \vec{r}\,'(t)dt$
    * Substitute the components of the parameterization of the curve in f or $\vec{v}$
    * Integrate normally with respect to t.

- This measures how much $\vec{v}$ "point in the same direction" as the curve C.
- The hardest part of this is often the parameterization. Make sure you know how to parameterize a line segment and a circle!

- New Differential Operators
  - Curl and Divergence are "differential operators" like partial derivatives and the gradient.
  - They both take vector fields, which is a new idea (partial derivatives and the gradient take functions instead.
  - If $\vec{v} = \langle P, Q, R \rangle$ then

$$div(\vec{v}) = \nabla \cdot \vec{v} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$
$$curl(\vec{v}) = \nabla \times \vec{v} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial y} - \frac{\partial Q}{\partial z} \right\rangle$$

    where we're using
$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

  - Notice, divergence takes vector fields and returns functions, while curl takes vector fields and returns vector fields. In fact, we have all of the combinations covered now:

$$\frac{\partial}{\partial x}: \text{functions} \to \text{functions}$$
$$\nabla: \text{functions} \to \text{vector fields}$$
$$div: \text{vector fields} \to \text{functions}$$
$$curl: \text{vector fields} \to \text{vector fields}$$

- Surface Integrals
  - Surface integrals allow you to integrate functions and vector fields over surfaces in $\mathbb{R}^3$.
  - We can compute

$$\iint_S \vec{v} \cdot d\vec{S} = \iint_S \vec{V} \cdot \hat{n}\ dS$$

    where $\hat{n}$ denotes the normal vector to the surface by
    * Parameterizing the surface S as $\vec{r}(u, v)$ with bounds $a \leq u \leq b, c \leq v \leq d$
    * Computing $\hat{n} = \vec{r}_u \times \vec{r}_v$
    * Substituting the components of the parameterization for $\vec{r}$ into $\vec{v}$

    * Integrating as a normal double integral.

  - The measures how much $\vec{v}$ "goes through" the surface, which is also called the flux.
  - Keep in mind, there are surfaces that are non-orientable, such as the Möbius strip or the Klein bottle, for which this cannot work. The problem is that there is no way to determine $\hat{n}$ across the whole surface.

4.2 Generalizations of the Fundamental Theorem

- Fundamental Theorem of Line Integration
  - The Fundamental Theorem of Line Integration (FTLI) is a generalization of the fundamental theorem from Calculus 1. It tells you that under certain conditions, you can just look at the endpoints of the curve, instead of all the work to parameterize and integrate.
  - Specifically, if C is a simple (doesn't self-intersect), smooth (no corners) curve and $\vec{v}$ is conservative with potential function f then

$$\int_C \vec{c} \cdot d\vec{s} = f(B) - f(A)$$

    where A is the starting point of C and B is the ending point of C.
  - In general, this is much easier than the standard method of computing line integrals, but it only works if $\vec{v}$ is conservative!
- Green's Theorem
  - Green's Theorem is a further generalization of the fundamental theorem. It relates the double integral over a region to the line integral around it's boundary.
  - Specifically, if R is a region in $\mathbb{R}^2$ that has a simple, smooth curve C as its boundary then

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C \langle P, Q \rangle \cdot d\vec{s}$$

    where C is taken to be positively-oriented (counterclockwise).
  - This allows you to take a line integral around a closed (starts and ends at the same place) curve and lets you rewrite it as a double integral, which can be much easier to compute.
  - Notice, if C is a closed loop (so that Green's Theorem applies) and also $\vec{v}$ is conservative (so that the FTLI applies) then the integral must be 0; you can see this by applying either theorem.

- There is an alternative way to write Green's Theorem:

$$\iint_R curl(\langle P, Q, 0\rangle) \cdot \hat{k}\ dA = \int_C \langle P, Q\rangle \cdot d\vec{s}$$

which is useful in understanding Stoke's Theorem.

- Stokes' Theorem
  - Stokes' Theorem further generalizes Green's Theorem from regions in $\mathbb{R}^2$ to surfaces in $\mathbb{R}^3$.
  - It states that for a surface S bounded by a simple, smooth curve C,

$$\iint_S curl(\vec{v}) \cdot \hat{n}\ dS = \int_C \vec{v}\ d\vec{s}$$

where the curve is positively-oriented.
  - Notice that for a surface in the xy-plane, $\hat{n} = \hat{k}$ and so this actually implies Green's Theorem!
  - One interesting consequence is that any two surfaces that share the same boundary will have the same integral!
- Divergence Theorem
  - This is our version of the Fundamental Theorem. It relates a triple integral over a volume to the double integral over its boundary.
  - If E is a volume in $\mathbb{R}^3$ and S is its (simple) boundary, then

$$\iiint_E div(\vec{v})\ dV = \iint_S \vec{v} \cdot \hat{n}\ dS$$

where $\hat{n}$ is chosen to be positively-oriented, which here means that it points outward from the surface.
  - This can significantly simplify our computations for surface integrals, by allowing us to rewrite them as triple integrals rather than by parameterizing the surface.
- The Moral
  - The goal of the last few sections, beyond introducing a slew of new "Fundamental Theorems" is to stress a deep idea in Calculus: Integration over a region can be reduced to integration over the boundary of the region. This was first expressed by the Fundamental Theorem of calculus (although it's not obvious, since $f(b) - f(a)$ doesn't look like an integral!) and it holds through to very arbitrary objects.
  - If you can understand and appreciate this idea, you've understood the point of multivariable calculus. The next step: try to rigorously prove what you've learned!

– If you're interested, look into the Generalized Stokes' Theorem

$$\int_{\Omega} d\omega = \int_{\partial\Omega} \omega$$

which contains every form of the Fundamental Theorem that you've seen thus far. Here Ω is an arbitrary "space," $\partial\Omega$ is its boundary, and ω is a "differential form," which is beyond the scope of this class, but somehow generalizes the idea of a vector field.