Witten's Laplacian and the Morse Inequalities

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Morse Theory

Morse Theory is the study of critical points of a smooth function $f : M \to \mathbb{R}$.

- A smooth manifold M is a topological manifold with compatible smooth atlas (in the following all manifolds are assumed to be n-dimensional, smooth, oriented, closed, and without boundary.)
- A critical point $q \in M$ of a smooth function $f : M \to \mathbb{R}$ is a zero of the differential df.
- The Hessian $H_f(q)$ of f at a critical point $q \in M$ is the matrix of second derivatives. (Independent of coordinate system at critical points.)

Morse Functions

- A smooth function $f : M \to \mathbb{R}$ is called Morse if its critical points are isolated and nondegenerate (that is, the Hessian of f is nonsingular.)
  - Remark: Nondegenerate critical points are necessarily isolated.
- The Morse index m_q of a critical point q is the dimension of the negative eigenspace of $H_f(q)$.
- The i-th Morse number M_i is the number of critical points with Morse index i.
  - Remark: The Morse numbers are invariant under diffeomorphism.

Betti Numbers

Associated to every smooth manifold is the sequence of Betti numbers $\beta_i, 0 \leq i \leq n$ defined as

$$\beta_i = \dim H^i_{dR}(M) = \dim \frac{\{\alpha \in \Omega^i : d\alpha = 0\}}{\{d\beta : \beta \in \Omega^{i-1}\}}$$

where Ω^i is the space of differential i-forms. This sequence is a topological invariant, and notably

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \beta_i$$

Morse Inequalities

The Weak Morse Inequalities are a classical result, proved using geometric techniques by Milnor [1].

Theorem (Weak Morse Inequalities)

Let $f : M \to \mathbb{R}$ be Morse. Then for any $0 \leq i \leq n$

$$\beta_i \leq M_i$$

and moreover

$$\chi(M) = \sum_{i=0}^{n} (-1)^i M_i$$

Morse Inequalities

Theorem (Polynomial Morse Inequalities)

Let $f : M \to \mathbb{R}$ be Morse. Then for any $t \in \mathbb{R}$ there exists a sequence of nonnegative integers Q_i such that

$$\mathcal{M}_t - \mathcal{P}_t := \sum_{i=0}^{n} M_i t^i - \sum_{i=0}^{n} \beta_i t^i = (1+t) \sum_{i=0}^{n-1} Q_i t^i$$

Theorem (Strong Morse Inequalities)

Let $f : M \to \mathbb{R}$ be Morse. Then for any $0 \leq k \leq n$

$$\sum_{i=0}^{k} (-1)^{i+k} \beta_i \leq \sum_{i=0}^{k} (-1)^{i+k} M_i$$

Edward Witten

In 1982, Edward Witten published a proof [2] of the Morse Inequalities, essentially using the idea of the flow generated by a Morse functions, with an intuition deriving from Quantum Mechanics. He was awarded a Fields Medal in 1990, partially for this work.

Supersymmetry

A Hilbert space $\mathcal{H}$ is called supersymmetric if there exists a decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ and maps

$$Q_1 : \mathcal{H}^+ \to \mathcal{H}^-$$
$$Q_2 : \mathcal{H}^- \to \mathcal{H}^+$$
$$H, (-1)^F : \mathcal{H} \to \mathcal{H}$$

that obey certain symmetry rules.

Supersymmetry

Notice that the space of differential forms is supersymmetric, splitting into even and odd forms, with

$$\Omega^* = \bigoplus_{i=0}^{n/2} \Omega^{2p} \oplus \bigoplus_{i=0}^{n/2-1} \Omega^{2p+1}$$

$$Q_1 = d + \delta$$

$$Q_2 = i(d - \delta)$$

$$H = \Delta = d\delta + \delta d$$

Witten Deformation

Witten generalizes this idea, conjugating d with the flow e^{tf} for $t \geq 0, f$ Morse.

$$d_t = e^{-tf} d e^{tf}$$
$$\delta_t = e^{tf} \delta e^{-tf}$$
$$\Delta_t = d_t \delta_t + \delta_t d_t$$

It is easily verifiable that Ω^* is still a supersymmetric space using these deformed operators.

Hodge Theory

To understand the motivation for the Witten Laplacian, we need to look to Hodge Theory.

Theorem (Hodge Theorem)

For $0 \leq i \leq n$, the maps

$$h_i : \ker \Delta^i \to H^i_{dR}(M)$$
$$\omega \mapsto [\omega]$$

are isomorphisms.

Corollary

For $0 \leq i \leq n$,

$$\beta_i = \dim \ker \Delta^i$$

Hodge Theory

The corollary is the starting point for analytic approaches to the Betti numbers. We will prove the Hodge Theorem using a heat flow argument, the following lemma will be necessary.

Lemma

For all smooth differential forms ω,

$$\Delta e^{-t\Delta}\omega = e^{-t\Delta}\Delta\omega$$

and

$$de^{-t\Delta}\omega = e^{-t\Delta}d\omega$$

The first claim follows from self-adjointness of Δ, while the second can be proved using the uniqueness of solutions to the heat equation in L^2.

Hodge Theory

Proof (Hodge Theorem).

Let $\{\omega_i\}_{i\in\mathbb{N}}$ be an orthonormal basis for Ω^p with $\Delta\omega_i = \lambda_i\omega_i$. This can be done since M is compact, and follows from the Spectral Theorem for compact, self-adjoint operators applied to the heat operator $e^{-t\Delta}$. Then

$$\lim_{t\to\infty} e^{-t\Delta}\omega = \lim_{t\to\infty}\sum_i a_i e^{-t\lambda_i}\omega_i = \sum_{i=0}^{N} a_i\omega_i$$

where $\{\omega_i, \ldots, \omega_N\}$ is an orthonormal basis for $\ker\Delta^p$. Thus as $t\to\infty, \omega$ flows to its harmonic component.

Hodge Theory

Proof (Hodge Theorem).

Then for any closed differential form ω,

$$\begin{aligned} e^{-t\Delta}\omega - \omega &= \int_0^t \partial_t(e^{-t\Delta}\omega)dt \\ &= d\left(-\int_0^t e^{-t\Delta}\delta\omega dt\right) \end{aligned}$$

so

$$e^{-t\Delta}\omega = \omega + d\left(-\int_0^t e^{-t\Delta}\delta\omega dt\right) \in [\omega]$$

which implies that heat flow preserves the cohomology class of a form.

Hodge Theory

Proof (Hodge Theorem).

We see that each cohomology class contains a harmonic form,

$$\lim_{t\to\infty} e^{-t\Delta}\omega = \omega - \lim_{t\to\infty} d\int_0^t e^{-t\Delta}\delta\omega\ dt = \omega - d\Delta^{-1}\delta\omega$$

(which is well-defined, after showing $\delta\omega$ is independent of $\ker\Delta$) , now finally we show that the form is unique. Assume there exist harmonic forms $\eta_1 \neq \eta_2$ with $[\eta_1] = [\eta_2]$ Then

$$\begin{aligned} \eta_1 &= \eta_2 + d\theta \\ \delta\eta_1 &= \delta\eta_2 + \delta d\theta \\ 0 &= \delta d\theta \end{aligned}$$

so

$$0 = \langle\theta, \delta d\theta\rangle = \langle d\theta, d\theta\rangle = \|d\theta\|^2$$

Witten Laplacian

Lemma

For any $t \geq 0$,

$$\beta_i = \dim \ker \Delta_t^i$$

Proof.

Observe that $d_t e^{-tf} = (e^{-tf} d e^{tf}) e^{-tf} = e^{-tf} d$, which implies that $e^{-tf} : \Omega^i \to \Omega^i$ is an isomorphism making the following diagram commute,

$$\begin{array}{ccccccc} \cdots & \xrightarrow{d} & \Omega^i & \xrightarrow{d} & \Omega^{i+1} & \xrightarrow{d} & \cdots \\ & & \big\downarrow{\scriptstyle e^{-tf}} & & \big\downarrow{\scriptstyle e^{-tf}} & & \\ \cdots & \xrightarrow{d_t} & \Omega^i & \xrightarrow{d_t} & \Omega^{i+1} & \xrightarrow{d_t} & \cdots \end{array}$$

so $\beta_i = \dim \ker \Delta^i = \dim \ker \Delta_t^i$. □

Witten Laplacian

Now we will see Witten's key insight: that the kernel of Δ_t^i is much simpler to understand as $t \to \infty$. It is necessary to expand the Witten Laplacian directly.

Witten Laplacian

For $\omega_1, \omega_2 \in \Omega^i$

$$\begin{aligned} d_t\omega_1 &= e^{-tf} d e^{tf} \omega_1 \\ &= e^{-tf}(e^{tf} d\omega_1 + te^{tf} df \wedge \omega_1) \\ &= (d + tdf\wedge)\omega_1 \end{aligned}$$

and

$$\begin{aligned} \langle \delta_t\omega_1, \omega_2 \rangle &= \langle \omega_1, d_t\omega_2 \rangle \\ &= \langle \omega_1, (d + tdf\wedge)\omega_2 \rangle \\ &= \langle \omega_1, d\omega_2 \rangle + \langle \omega_1, tdf \wedge \omega_2 \rangle \\ &= \langle \delta\omega_1, \omega_2 \rangle + \langle t\iota_{\nabla f}\omega_1, \omega_2 \rangle \\ &= \langle (\delta + t\iota_{\nabla f})\omega_1, \omega_2 \rangle \end{aligned}$$

Witten Laplacian

Expanding Δ_t,

$$\begin{aligned}
\Delta_t &= d_t\delta_t + \delta_t d_t \\
&= (d + tdf\wedge)(\delta + t\iota_{\nabla f}) + (\delta + t\iota_{\nabla f})(d + tdf\wedge) \\
&= d\delta + tdf\wedge\delta + td\iota_{\nabla f} + t^2 df\wedge\iota_{\nabla f} \\
&\qquad + \delta d + t\iota_{\nabla f}d + t\delta df\wedge + t^2\iota_{\nabla f}df\wedge \\
&= \Delta + t^2\|df\|^2 + th
\end{aligned}$$

where

$$h = \mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f}$$

Witten Laplacian

Then as t becomes large, $\omega \in \ker \Delta_t$ implies that ω can be nonzero only on small neighborhoods of the critical points of f. We will now consider a neighborhood U_q of a critical point q of f, and compute Δ_t in a local coordinate system on U_q.

Morse Lemma

We will use the Morse Lemma to provide a coordinate system.

Theorem (Morse Lemma)

Let q be an isolated, nondegenerate critical point for $f \in C^\infty(M, \mathbb{R})$. Then there exists a coordinate system $\{x_1, \ldots, x_n\}$ on a neighborhood U_q of q such that for $x = (x_1, \ldots, x_n) \in U_q$,

$$f(x) = f(q) - \sum_{i=1}^{m_q} x_i^2 + \sum_{i=m_q+1}^{n} x_i^2$$

where m_q is the Morse index of q.

Morse Lemma

To prove the Morse Lemma, we follow the following outline:

1. Prove Hadamard's Lemma (1st order Taylor expansion).
2. Show that if $f(x) = f(q) - \sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^{n} x_i^2$, then $\lambda = m_q$.
3. Choose a coordinate system $x_1(q) = \cdots = x_n(q) = 0$, and apply Hadamard's Lemma twice (using the fact that $df(q) = 0$.)
4. Argue inductively on the coordinates x_i.

Local Coordinate Expansion

In the coordinate system provided by the Morse Lemma,

$$\|df\|^2 = 4\sum_{i=1}^{n} x_i^2,$$

$$\nabla f = -\sum_{i=1}^{m_q} 2x_i \frac{\partial}{\partial x_i} + \sum_{i=m_q+1}^{n} 2x_i \frac{\partial}{\partial x_i},$$

and by choosing the metric in local coordinates to be $g = \sum_{i=1}^{n}(dx^i)^2$ at q,

$$\Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

Local Coordinate Expansion

A (long) computation for $h = \mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f}$ gives

$$h = 2\sum_{i=1}^{n} \eta_i [dx^i, \iota_{\frac{\partial}{\partial x_i}}]$$

where

$$\eta_i = \begin{cases} -1 & i \leq m_q, \\ 1, & i > m_q. \end{cases}$$

Local Coordinate Expansion

Thus on U_q we can approximate Δ_t by,

$$
\begin{aligned}
H_{q,t} &= \sum_{i=1}^{n} -\frac{\partial^2}{\partial x_i^2} + 4t^2 x_i^2 + 2\eta_i t[dx^i \wedge, \iota_{\frac{\partial}{\partial x_i}}] \\
&= \sum_{i=1}^{n} J_i + 2tK_i
\end{aligned}
$$

Where $J_i := -\frac{\partial^2}{\partial x_i^2} + 4t^2 x_i^2$ and $K_i := \eta_i [dx^i, \iota_{\frac{\partial}{\partial x_i}}]$.

Local Coordinate Expansion

Notice, for a differential form $\omega = f_I dx^I$ (with I a multiindex,)

$$K_i\omega = \begin{cases} -\omega & (i \leq m_q \text{ and } i \in I) \text{ or } (i > m_q \text{ and } i \notin I) \\ \omega & \text{otherwise} \end{cases}$$

so $K_i = \pm 1 \implies [J_i, K_i] = 0$ which implies that they can be simultaneously diagonalized.

Local Coordinate Approximation

Proposition (Kernel of $H_{q,t}$)

For any $q \in Cr(f)$, the map $H_{q,t} : U_q \to \mathbb{R}$ defined in the local coordinates $\{x_i\}$ given by the Morse lemma on the neighborhood U_q of q by

$$H_{q,t} = \sum_{i=1}^{n} -\frac{\partial^2}{\partial x_i^2} + 4t^2 x_i^2 + 2\eta_i t[dx^i \wedge, \iota_{\frac{\partial}{\partial x_i}}]$$

has kernel of dimension one, and is generated by the eigenform

$$e^{-t|x|^2} dx^1 \wedge \cdots \wedge dx^{m_q}.$$

and moreover all of the nonzero eigenvalues of $H_{q,t}$ are greater than Ct for some fixed $C > 0$.

Eigenvalues of the Quantum Harmonic Oscillator

The operator J_i is a scaling of the simple quantum harmonic oscillator from physics. It's spectrum is well-known.

Proposition

The eigenvalues of J_i are precisely $2t(1+2j)$ for non-negative integers j. Moreover, the $2t$-eigenfunction of J_i is

$$e^{-tx_i^2}$$

Eigenvalues of the Quantum Harmonic Oscillator I

To determine the eigenvalues of the quantum harmonic oscillator, argue using the 'Dirac Ladder Operator' method:

1. Define $p = -i\frac{\partial}{\partial x_i}$ and $a = \sqrt{t}\left(x_i + \frac{i}{2t}p\right)$ so that

$$J_i = 4t^2x_i^2 + p^2$$

2. Then

$$2t(1 + 2a^\dagger a) = J_i$$

so an eigenvalue of $N = a^\dagger a$ is an eigenvalue of J_i.

3. Show that

$$Naf_\lambda = (\lambda - 1)af_\lambda$$
$$Na^\dagger f_\lambda = (\lambda + 1)a^\dagger f_\lambda.$$

Eigenvalues of the Quantum Harmonic Oscillator II

4. Argue that eigenvalues must be nonnegative since

$$\begin{aligned}\lambda\langle f_\lambda, f_\lambda\rangle &= \langle f_\lambda, a^\dagger a f_\lambda\rangle \\ &= \|a f_\lambda\|^2 \geq 0\end{aligned}$$

5. Argue that if λ is not a nonnegative integer, applying a sufficiently many times to f_λ would result in a function with negative eigenvalue, a contradiction.
6. Conclude the first claim using $2t(1+2N) = J_i$.
7. Finally, show that $e^{-tx_i^2}$ is the $2t$-eigenfunction of J_i directly, solving $Naf_0 = 0$.

Local Coordinate Approximation

Now we can prove the proposition. For an eigenform ω of $H_{q,t}$,

$$H_{q,t}\omega = t\left(2\sum_{i=1}^{n}(1+2j+K_i)\right)\omega$$

so if $\omega = g(x)dx^I \in \ker H_{q,t}$ is nontrivial, then it must be that $j=0$ which forces $g(x) = e^{-t|x|^2}$ and also $i \in I$ if and only if $i \leq m_q$. In conclusion,

$$\omega = e^{-t|x|^2}dx^1 \wedge \cdots \wedge dx^{m_q}$$

generates the kernel of $H_{q,t}$.

Physical Intuition

There is now a good intuition as to why the Weak Morse Inequalities should hold.

- For each $q \in Cr(f)$ with $m_q = p$, the kernel of $H_{q,t}$ is generated by a single p-form locally.
- There will be precisely M_p critical points of f with p-forms generating the kernel of $H_{q,t}|_\Omega^p$ locally.
- Globally it seems reasonable that for $\omega \in \ker \Delta_t^p$ it must be that $\omega \in H_{q,t}$ so $M_p \geq \dim \ker \Delta_t^p = \beta_p$.
- Witten argues along these lines. Here is presented a justification by global analysis of the low-lying eigenvalues of Δ_t^p, adapted from [3].

Proof of the Weak Morse Inequalities

Denote by $E_t^p(c)$ the eigenspace of Δ_t^p with eigenvalues in the interval $[0, c]$. The following key theorem will give the Weak Morse Inequalities as a corollary.

Theorem (Key Theorem)

For any $c > 0$, there exists a $t_0 > 0$ such that for any $t > t_0$,

$$\dim E_t^p(c) = M_p$$

where M_p is the p-th Morse number, $0 \le p \le n$.

Corollary (Weak Morse Inequalities)

For $0 \le p \le n$,

$$\beta_p \le M_p$$

Proof of the Weak Morse Inequalities

To prove the key theorem, we will need estimates on Sobolev norms. Without loss of generality assume U_q is an open ball centered at critical point q with radius $4a$, and choose $\gamma \in C^\infty(\mathbb{R}, [0,1])$ to be such that

$$\gamma(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > 2a \end{cases}$$

Proof of the Weak Morse Inequalities

Define

$$\alpha_{q,t} = \left\| \gamma(|x|)e^{-t|x|^2} \right\|_0^2 = \int_{U_q} \gamma(|x|)^2 e^{-2t|x|^2} \; dx^1 \wedge \cdots \wedge dx^n$$

$$\rho_{q,t} = \frac{\gamma(|x|)}{\sqrt{\alpha_{q,t}}} e^{-t|x|^2} dx^1 \wedge \cdots \wedge dx^{m_q}$$

The $\rho_{q,t}$ will have unit length, and are motivated by the local generators for the kernel of $H_{q,t}$. Denote by E_t the vector space genersated by the $\rho_{q,t}$ for $q \in Cr(f)$.

Projection Lemma

The following lemma estimating the orthogonal projections $P_t(c)\colon H^0(M) \to E_t(c)$ will allow us to prove the key theorem.

Lemma

There exist constants $C, t_3 > 0$ such that for any $t \geq t_3$ and any $\sigma \in E_t$,

$$\|P_t(c)\sigma - \sigma\|_0 \leq \frac{C}{t}\|\sigma\|_0$$

Deformed Witten Operator

Denote by D_t the 'deformed Witten operator.'

$$D_t = d_t + \delta_t$$

and observe that

$$\Delta_t = d_t\delta_t + \delta_t d_t = (d_t + \delta_t)(d_t + \delta_t) = D_t^2$$

since $d_t^2 = \delta_t^2 = 0$.

Deformed Witten Operator

Since the $\rho_{q,t}$ are compactly supported, E_t is a finite dimensional subspace of $H^0(M)$, and so there is an orthogonal decomposition

$$H^0(M) = E_t \oplus E_t^\perp$$

with projections

$$p_t : H^0(M) \to E_t$$
$$p_t^\perp : H^0(M) \to E_t^\perp$$

Proof of Key Theorem

Proof (Key Theorem).

By the lemma, when t is large enough the $P_t(c)\rho_{q,t}$ will be linearly independent, so

$$\dim E_t(c) \geq \dim E_t.$$

Assume for contradiction that $\dim E_t(c) > \dim E_t$. Then there must be a nonzero $s \in E_t(c)$ orthogonal to $P_t(c)E_t$. That is

$$\langle s, P_t(c)\rho_{q,t}\rangle_0 = 0$$

for all $q \in Cr(f)$.

Proof of Key Theorem

Proof (Key Theorem).

Then we can write the projection

$$\begin{aligned} p_t s &= \sum_{q \in Cr(f)} \langle s, \rho_{q,t} \rangle_0 \rho_{q,t} \\ &= \sum_{q \in Cr(f)} \langle s, \rho_{q,t} \rangle_0 (\rho_{q,t} - P_t(c)\rho_{q,t}) \\ &\quad + \sum_{q \in Cr(f)} \langle s, \rho_{q,t} - P_t(c)\rho_{q,t} \rangle_0 P_t(c)\rho_{q,t} \end{aligned}$$

so by the lemma

$$\|p_t s\|_0 \leq \frac{C}{t} \|s\|_0.$$

Proof of Key Theorem

Proof (Key Theorem).

Then

$$\|p_t^\perp s\|_0 \geq \|s\|_0 - \|p_t s\|_0 \geq C'\|s\|_0$$

and then using the proposition,

$$\begin{aligned} CC'\sqrt{t}\|s\|_0 \leq \|D_t p_t^\perp s\|_0 &\leq \|D_t s\|_0 + \|D_t p_t s\|_0 \\ &\leq \|D_t s\|_0 + \frac{1}{t}\|s\|_0. \end{aligned}$$

Rearranging,

$$\|D_t s\|_0 \geq \frac{CC' t^{3/2} - 1}{t}\|s\|_0$$

which as $t \to \infty$ contradicts $s \in E_t(c)$.

Proof of Key Theorem

Proof (Key Theorem).

Now

$$\dim E_t(c) = \dim E_t = \sum_{i=0}^{n} M_i$$

and $E_t(c)$ is generated by $P_t(c)\rho_{q,t}$. Let Q_i denote the projection $H^0(M) \to L^2\Omega^i$. We have that

$$\Delta_t Q_i s = Q_i \Delta_t s = \mu^2 Q_i s$$

so that $Q_i s$ is a μ^2-eigenform of Δ_t. We wish to show that for t large enough, $\dim Q_i E_t(c) = M_i$.

Proof of Key Theorem

Proof (Key Theorem).

By the lemma, for t large enough

$$\|Q_{m_q}P_t(c)\rho_{q,t} - \rho_{q,t}\|_0 \leq \frac{C}{t}$$

thus the forms $Q_{m_q}P_t(c)\rho_{q,t}$ are linearly independent and $\dim Q_iE_t(c) \geq M_i$. However,

$$\sum_{i=0}^{n} \dim Q_iE_t(c) \leq \dim E_t(c) = \sum_{i=0}^{n} M_i$$

forcing $\dim Q_iE_t(c) = M_i$, and completing the proof. □

Polynomial Morse Inequalities I

Witten proves the Polynomial Morse Inequalities. The Strong Morse Inequalities follow by equivalence. We outline the proof of the Polynomial Morse Inequalties:

1. Let $C_p(f)$ be the free abelian group generated by the critical points $q \in M$ with Morse index $m_q = p$. Denote by d_t^p the map $C_p(f) \to C_{p+1}(f)$ determined by d_t, identifying critical points with the associated eigenforms of Δ_t. Consider the Morse-Smale-Witten chain complex

$$0 \to C_1(f) \to \cdots \to C_n(f) \to 0.$$

Polynomial Morse Inequalities II

2. The sequence

$$0 \to \ker d_t^p \to C_p(f) \xrightarrow{d_t^p} \operatorname{im} d_t^p \to 0$$

is exact, so

$$M_p = \operatorname{rank} C_p(f) = \operatorname{rank} \ker d_t^p + \operatorname{rank} \operatorname{im} d_t^p$$

3. The sequence

$$0 \to \operatorname{im} d_t^{p-1} \to \ker d_t^p \to H_k(C_*(f), d_t^*) \to 0$$

is also exact, so

$$\beta_p = \operatorname{rank} H_k(C_*(f), d_t^*) = \operatorname{rank} \ker d_t^p - \operatorname{rank} \operatorname{im} d_t^{p-1}$$

Polynomial Morse Inequalities III

4 Then letting $Q_p = M_p - \operatorname{rank}\ker d_t^p \geq 0$,

$$\begin{aligned}\mathcal{M}_t - \mathcal{P}_t &= \sum_{p=0}^{n} (\operatorname{rank}\ker d_t^p + \operatorname{rank}\operatorname{im} d_t^p) t^p \\ &\quad - \sum_{p=0}^{n} (\operatorname{rank}\ker d_t^p - \operatorname{rank}\operatorname{im} d_t^{p-1}) t^p \\ &= (1+t) \sum_{p=0}^{n-1} (M_p - \operatorname{rank}\ker d_t^p) t^p \\ &= (1+t) \sum_{p=0}^{n-1} Q_p t^p\end{aligned}$$

Strong Morse Inequalities I

Theorem

The Strong and Polynomial Morse Inequalities are equivalent.

This can be proved as follows:

1. Assume the Strong Inequalities. Then

$$\mathcal{M}_{-1} = \sum_{i=0}^{n} (-1)^i M_i = \sum_{i=0}^{n} (-1)^i \beta_i = \mathcal{P}_{-1}$$

   implies that $\mathcal{M}_t - \mathcal{P}_t$ is divisible by $(1+t)$.

2. Then for some $Q_i \in \mathbb{Z}$,

$$\mathcal{M}_t - \mathcal{P}_t = (1+t) \sum_{i=0}^{n-1} Q_i t^i$$

Strong Morse Inequalities II

3. Arguing by induction using the Strong Inequalities, we can show that the Q_i must be nonnegative, proving the Polynomial Inequalities.
4. Assume the Polynomial Inequalities. Then by induction for $k \in \{0, 1, \ldots, n-1\}$,

$$\sum_{i=0}^{k} (-1)^{i+k} M_i = \sum_{i=0}^{k} (-1)^{i+k} \beta_i + Q_k$$

5. Letting $t = -1$,

$$\sum_{i=0}^{n} (-1)^{i} M_i = \sum_{i=0}^{n} (-1)^{i} \beta_i$$

completing the equivalence.

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