The Horizontal Einstein Property for H-Type Foliations

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Basic Definitions

Let $\mathbb{M}$ be a smooth manifold. We say that

- $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$ is a sub-Riemannian manifold if $\mathcal{H}$ is a constant rank, bracket generating subbundle of $T\mathbb{M}$, and $g_{\mathcal{H}}$ is a inner product on $\mathcal{H}$.
- $(\mathbb{M}, \mathcal{H}, g)$ is a sub-Riemannian manifold with metric preserving complement if $(\mathbb{M}, g)$ is a Riemannian manifold and the metric orthogonally splits as $g = g_{\mathcal{H}} \oplus g_{\mathcal{V}}$ such that $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$ is a sub-Riemannian manifold. We denote by $\mathcal{V}$ the orthogonal complement of $\mathcal{H}$ by g.

Gromov-Hausdorff Convergence

For a sub-Riemannian manifold with metric preserving complement $(\mathbb{M}, \mathcal{H}, g)$ we define the canonical variation of the metric

$$g^{\epsilon} = g_{\mathcal{H}} + \frac{1}{\epsilon} g_{\mathcal{V}}$$

which motivates our interest, since in the Gromov-Hausdorff sense

$$(\mathbb{M}, \mathcal{H}, g^{\epsilon}) \xrightarrow{\epsilon \to 0^+} (\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$$

Hladky-Bott Connection

Theorem (Hladky '12 [4])

There exists a unique metric connection ∇ on $(\mathbb{M}, \mathcal{H}, g)$ such that

1. *$\mathcal{H}$ and $\mathcal{V}$ are ∇-parallel,*
2. *The torsion T of ∇ satisfies*
   - $T(\mathcal{H}, \mathcal{H}) \subset \mathcal{V}$,
   - $T(\mathcal{V}, \mathcal{V}) \subset \mathcal{H}$
3. *For every $X, Y \in \Gamma(\mathcal{H}), Z, V \in \Gamma(\mathcal{V})$,*
   - $\langle T(X, Z), Y \rangle_{\mathcal{H}} = \langle T(Y, Z), X \rangle_{\mathcal{H}}$
   - $\langle T(Z, X), V \rangle_{\mathcal{V}} = \langle T(V, X), Z \rangle_{\mathcal{V}}$.

This is called the Hladky-Bott connection.

Hladky-Bott Connection

We can determine ∇ using the Levi-Civita connection ∇^g as

$$\nabla_X Y = \begin{cases} \pi_{\mathcal{H}} \nabla^g_X Y & X, Y \in \Gamma(\mathcal{H}) \\ \pi_{\mathcal{H}}[X, Y] + A_X Y & Y \in \Gamma(\mathcal{H}), X \in \Gamma(\mathcal{V}) \\ \pi_{\mathcal{V}}[X, Y] + A_X Y & Y \in \Gamma(\mathcal{V}), X \in \Gamma(\mathcal{H}) \\ \pi_{\mathcal{V}} \nabla^g_X Y & X, Y \in \Gamma(\mathcal{V}) \end{cases}$$

where the tensor A is defined by

$$\langle A_X Y, Z \rangle = \frac{1}{2}\left((\mathcal{L}_{X_{\mathcal{V}}} g)(Y_{\mathcal{H}}, Z_{\mathcal{H}}) + (\mathcal{L}_{X_{\mathcal{H}}} g)(Y_{\mathcal{V}}, Z_{\mathcal{V}})\right)$$

J Map

On $(\mathbb{M}, \mathcal{H}, g)$ we can associate to each vector field $Z \in \Gamma(T\mathbb{M})$ an endomorphism J_Z of $T\mathbb{M}$ defined by

$$\langle J_Z X, Y \rangle = \langle Z, T(X, Y) \rangle$$

If $\mathcal{V}$ is integrable, the only nontrivial case is $J_{\mathcal{V}}(\mathcal{H}) \subset \mathcal{H}$ and so we can consider

$$J \colon \mathcal{V} \to \mathrm{End}(\mathcal{H}), \quad Z \mapsto J_Z$$

H-type Foliations

Definition

Let $(\mathbb{M}, \mathcal{H}, g)$ be a sub-Riemannian manifold with metric preserving complement. We say that $(\mathbb{M}, \mathcal{H}, g, J)$ is an <u>H-type foliation</u> if

1. $\mathcal{V}$ is integrable, and
2. for all $X, Y \in \Gamma(\mathcal{H}), Z \in \Gamma(\mathcal{V})$,

$$\langle J_Z X, J_Z Y \rangle_{\mathcal{H}} = \|Z\|^2 \langle X, Y \rangle_{\mathcal{H}}$$

Parallel Torsion

We also refine the definition of H-type foliations based on the behavior of derivatives of the Hladky-Bott torsion T.

1. If $\delta_{\mathcal{H}} T = 0$ we say $\mathbb{M}$ is of Yang-Mills type,
2. If $\nabla_{\mathcal{H}} T = 0$ we say $\mathbb{M}$ has horizontally parallel torsion, and
3. If $\nabla T = 0$ we say $\mathbb{M}$ has completely parallel torsion.

Examples of H-type foliations

rk$(\mathcal{V})$	**rk**$(\mathcal{H})$	$\mathbb{M}$	Torsion
any	2k	H-type Heisenberg Group	C. Parallel
1	2k	Mixed K-contact	Yang-Mills
		Sasakian/Hopf fibration	C. Parallel
2	4k	Salamon Twistor Spaces	H. Parallel
3	4k	Mixed 3K-contact	Yang-Mills
		3-Sasakian/Quaternionic Hopf	H. Parallel
		Torus Bundles over HK	C. Parallel
4,5,7	8k	Grassmannian Type	H. Parallel
7	8	Octonionic Hopf	H. Parallel

Example: Hopf Fibration

Consider $\mathbb{S}^{2n+1}$ foliated as

$$\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1} \xrightarrow{\pi} \mathbb{C}P^n$$

Defining the horizontal distribution as

$$\mathcal{H} = d\pi^{-1}(T\mathbb{C}P^n)$$

the space $(\mathbb{S}^{2n+1}, \mathcal{H}, g)$ is an H-type foliation; the vertical space is determined by

$$\mathcal{V}_x = \mathcal{H}_x^{\perp} \cong T_x\mathbb{S}^1$$

so we have $m = \dim \mathcal{V} = 1$.

Example: Twistor Spaces

Let $(\mathbb{M}, g)$ be a $4n$-dimensional $(n \geq 2)$ quaternionic-Kähler manifold, and fix a quaternionic structure E spanned by $\mathcal{I}, \mathcal{J}, \mathcal{K} \in \mathrm{End}(T\mathbb{M})$. Choosing a metric on E so that $\mathcal{I}, \mathcal{J}, \mathcal{K}$ are orthonormal, we define the twistor space over $\mathbb{M}$ to be the unit sphere bundle of E. In this case, we have

- $\mathcal{H}_x \cong T_x\mathbb{M}$,
- $\mathcal{V}_x \cong \mathbb{C}P^1$.

Here $m = \dim \mathcal{V} = 2$ and there is a quaternionic structure induced by $\mathcal{V}$ acting on $\mathcal{H}$.

Dimensional Restrictions

Lemma

Denote $m = \mathbf{rk}(\mathcal{V}), n = \mathbf{rk}(\mathcal{H})$. *Then*

1. $m \leq n - 1$,
2. $m = n - 1$ *implies* $n = 2, 4,$ *or* 8,
3. $n = 2k$, *and furthermore*
   - *if* $m \geq 2$ *then* $n = 4k$,
   - *if* $m \geq 4$ *then* $n = 8k$.

Clifford Structures

Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation with $Z_i, Z_j \in \mathcal{V}$, then

$$J_{Z_i} J_{Z_j} + J_{Z_j} J_{Z_i} = -2\langle Z_i, Z_j \rangle Id_{\mathcal{H}}$$

and so we can extend J in the natural way to

$$J \colon Cl(\mathcal{V}) \to \mathrm{End}(\mathcal{H})$$

There is a classification of such Clifford structures over Riemannian manifolds. (A. Moroianu, U. Semmelmann '11 [6].)

Parallel Horizontal Clifford Structures

Definition

Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation with horizontally parallel torsion. Then if there exists a map

$$\Psi : \mathcal{V} \times \mathcal{V} \to Cl_2(\mathcal{V})$$

such that

$$(\nabla_{Z_1} J)_{Z_2} = J_{\Psi(Z_1, Z_2)}$$

for all $Z_1, Z_2 \in \Gamma(\mathcal{V})$ then we say that $\mathbb{M}$ has a parallel horizontal Clifford structure.

Parallel Horizontal Clifford Structures

Lemma (Baudoin, Grong, Rizzi, & M. '18 [1])

Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation with parallel horizontal Clifford structure. There exists $\kappa \in \mathbb{R}$ such that

$$\Psi(Z_1, Z_2) = \kappa(Z_1 \cdot Z_2 + \langle Z_1, Z_2 \rangle)$$

for all $Z_1, Z_2 \in \Gamma(\mathcal{V})$; moreover the sectional curvature of the leaves associated to $\mathcal{V}$ is constant and equal to κ^2.

Einstein Manifolds

Definition

Let $(\mathbb{M}, g)$ be a Riemannian manifold. We say that it is an Einstein manifold if there exists some constant $\lambda \in \mathbb{R}$ such that

$$\langle X, Y \rangle = \lambda \, \mathrm{Ric}(X, Y)$$

for all $X, Y \in \Gamma(T\mathbb{M})$.

Einstein Manifolds

- Originate in physics, as solutions to the Einstein field equations in vacuum.
- Mathematically, these are of great interest as model spaces.
- Ideal for computation, while still including a large class of structures.
- Include Euclidean space, complex projective spaces, and Calabi-Yau manifolds

Horizontal Ricci Curvature

Let $(\mathbb{M}, \mathcal{H}, g^\epsilon)$ be a sub-Riemannian manifold with metric preserving complement equipped with the canonical variation. Unfortunately, in the limit $\epsilon \to 0^+$ the Ricci curvature associated to ∇ is not well defined. c.f. (Baudoin, Kim, Wang '16 [3]).

Horizontal Ricci Curvature

We define the horizontal and vertical Riemann curvature tensors by

$$R_{\mathcal{H}}(X, Y)Z = R^{\nabla}(X_{\mathcal{H}}, Y_{\mathcal{H}})Z_{\mathcal{H}},$$
$$R_{\mathcal{V}}(X, Y)Z = R^{\nabla}(X_{\mathcal{V}}, Y_{\mathcal{V}})Z_{\mathcal{V}},$$

from which it will follow that

$$R^{\nabla}(X, Y)Z = R_{\mathcal{H}}(X, Y)Z + R_{\mathcal{V}}(X, Y)Z + (\nabla_Z T)(X, Y)$$

Horizontal Ricci Curvature

Definition

The horizontal Ricci curvature $\mathrm{Ric}_{\mathcal{H}}$ is the horizontal trace of $R_{\mathcal{H}}$:

$$\mathrm{Ric}_{\mathcal{H}}(X, Y) = \sum_{a=1}^{n} R_{\mathcal{H}}(X, X_a, Y, X_a)$$

where $X_1, \cdots, X_n$ is a local orthonormal frame for $\mathcal{H}$.

The Horizontal Einstein Property

We can now define an analog of the Einstein property in our setting:

Definition

Let $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$ be a sub-Riemannian manifold. We say that $\mathbb{M}$ has the horizontal Einstein property if there exists some constant $\lambda \in \mathbb{R}$ such that

$$\langle X, Y \rangle_{\mathcal{H}} = \lambda \, \mathrm{Ric}_{\mathcal{H}}(X, Y)$$

for all $X, Y \in \Gamma(\mathcal{H})$.

Main Theorem

Theorem (Baudoin, Grong, Rizzi, & M. '18 [1] [2])

Let $(\mathbb{M}, \mathcal{H}, g)$ *be an H-type foliation with parallel horizontal Clifford structure, with* $m \geq 2$. *Then* $\mathbb{M}$ *is horizontally Einstein with*

$$\lambda = \kappa \left(\frac{n}{4} + 2m - 4 \right)$$

Moreover,

- *if* $\kappa \neq 0$ *then* $\mathbf{Hol}(\mathcal{H}) \subset Sp(1)Sp(n/4)$,
- *if* $\kappa = 0$ *then* $\mathbf{Hol}(\mathcal{H}) \subset Sp(n/4)$.

Case $2 \leq m \neq 3$

The proof follows the approach of Moroianu and Semmelmann,

1. Show
$$[R_{\mathcal{H}}(X,Y), J_{ij}] = \sum_s \sigma_{si}(X,Y) J_{sj}$$
where $J_{ij} \in Cl^0(\mathcal{V})$,
2. Express $\mathrm{Ric}_{\mathcal{H}}$ in terms of σ_{ij},
3. Solve explicitly, using
$$\sigma_{ij}(X,Y) = \langle (\nabla_{Z_j} T)(X,Y), Z_i \rangle$$

Case $m = 3$

In when $\mathbf{rk}(\mathcal{V}) = 3$, there is a splitting

$$Cl^0(\mathcal{V}) \cong \mathbb{H} \oplus \mathbb{H}$$

which causes the argument to fail. In this case we must work directly on $Cl(\mathcal{V})$, analogously to the proof (Ishihara '74 [5]) that quaternion-Kähler manifolds are Einstein.

Hol($\mathcal{H}$)

We extend and apply Ambrose-Singer.

1. Extend Ambrose-Singer theorem to $\mathcal{H}$ when $\nabla_{\mathcal{H}} T = 0$.
2. Apply to see that **Hol**($\mathcal{H}$) is determined by the endomorphisms of $R_{\mathcal{H}}$.
3. Conclude, using that the $[R_{\mathcal{H}}(X, Y), J_{ij}]$ preserve a quaternionic structure.

Further Work

This is a component of forthcoming work with Fabrice Baudoin, Erlend Grong, and Luca Rizzi. We are able to prove in this setting

1. Bonnet-Meyers diameter bounds,
2. sub-Hessian and sub-Laplacian comparisons,
3. curvature dimension inequalities and measure contraction properties.

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Thank you for your attention!