H-type foliations

Fabrice Baudoin\textsuperscript{1}, Erlend Grong\textsuperscript{2}, Gianmarco Molino\textsuperscript{1}, Luca Rizzi\textsuperscript{3}

\textsuperscript{1}Univ. of Connecticut, Storrs, CT, USA; \textsuperscript{2}Univ. Paris-Sud, Gif-sur-Yvette, France; \textsuperscript{3}Univ. Grenoble Alpes, Grenoble, France

Abstract

With a view toward sub-Riemannian geometry, we introduce and study H-type foliations. These structures are natural generalizations of K-contact manifolds which encompass as special cases K-contact manifolds, invariant spaces, K-contact manifolds and H-type groups. Under an horizontal Ricci curvature lower bound, we prove on those structures sub-Riemannian diameter upper bounds and first eigenvalue estimates for the sub-Laplacian. Then, using a result by Moroianu-Semmelmann, we classify the H-type foliations that carry a parallel horizontal Clifford structure. Finally, we prove an horizontal Einstein property and compute the horizontal Ricci curvature of those structures in codimension more than 2.

Background

A sub-Riemannian manifold is a smooth manifold equipped with a bracket generating distribution $H \subset TM$ and a fiber inner product $g_H$ on $H$. We require the existence of a transverse totally geodesic and integrable complement $V$. From this, we introduce and study a new class of sub-Riemannian manifolds generalizing the H-type groups introduced by Kaplan in [5]. We call such manifolds H-type sub-Riemannian manifolds.

Proposition 1. (H-type foliations) There exists a unique metric connection $\nabla$ on $M$, called the Bott connection of the foliation, such that:

1. $H$ and $V$ are $\nabla$-parallel, i.e. for every $X \in \Gamma(H)$, $Y \in \Gamma(TM)$ and $Z \in \Gamma(V)$,
   \[ \nabla_X Y \in \Gamma(H), \ 
   \nabla_X Z \in \Gamma(V). \tag{1} \]

2. The torsion $T$ of $\nabla$ satisfies
   \[ T(H, V, H) = 0, \ 
   T(H, V, V) = 0. \tag{2} \]

This connection is better suited to the study of the foliation structure, as it preserves the horizontal and vertical bundles.

Claim 2. (Kaplan’s map) For $X, Y \in \Gamma(V)$ there exists a unique skew-symmetric fiber endomorphism $\sigma(X, Y) : \Gamma(H) \to \Gamma(H)$ such that
\[ g_H(\sigma(X, Y)Z, W) = g_H(Z, \nabla_X W) \]
for every $Z, W \in \Gamma(H)$, where $T(H, V, W) = -V \times \nabla_X H$ is the torsion of the Bott connection.

Definition 3. (H-type sub-Riemannian Manifold) Let $M$ be a smooth, oriented, connected, manifold of dimension $n = m + 2$ equipped with a Riemannian foliation with bundle-like metric $g$ and totally geodesic $m$-dimensional leaves.

\[ \langle JX, JY \rangle = |J|^{-2} \langle X, Y \rangle \tag{4} \]

we say that $(M, g)$ is an H-type foliation.

Due to their symmetries, H-type sub-Riemannian manifolds provide an ideal framework to develop a program reducing the study of global geometric, metric, or analytic properties of the ambient space to the study of local sub-Riemannian curvature type invariants.

Main Results

Yang-Mills Property

We show that all H-type foliations are Yang-Mills; as a consequence, the sub-Laplacian of an H-type foliation satisfies a simple Bochner’s type theorem and the validity of the generalized curvature dimension inequality is only controlled by the horizontal Ricci curvature of the Bott connection.

Proposition 4. Let $(M, g)$ be an H-type foliation such that $Ric_H \geq K g_H$ with $K \in \mathbb{R}$. Then $(M, (\mathbb{R}, g))$ satisfies the generalized curvature dimension inequality $CD(K, \mathbb{R}, n, m)$, i.e. for every $f \in \mathcal{C}^0(M)$ and $\epsilon > 0$, one has the following Bochner’s type inequality:
\[ \frac{1}{2} \langle \Delta_{CD} f \rangle = \frac{1}{2} \langle \Delta_H f \rangle = \frac{1}{2} \langle \mathcal{L}_H f \rangle - \frac{1}{2} \epsilon \langle |\nabla_{\mathcal{L}_H} f| \rangle + \frac{1}{2} \epsilon \langle |\nabla_{\mathcal{L}_H} f| \rangle \]
\[ \geq \frac{1}{2} \langle \Delta_H f \rangle + \left( 1 - \epsilon \right) \langle |\nabla f| \rangle \]
\[ \geq \frac{1}{2} \langle \Delta_H f \rangle + \left( 1 - \epsilon \right) \langle \frac{1}{2} |\nabla f| \rangle. \tag{5} \]

The consequences of the generalized curvature dimension inequality have been extensively studied recently (see [1, 2, 3]), in our setting we will have

Corollary 5. Let $(M, g)$ be a complete H-type foliation with $Ric_H \geq K g_H$ with $K \in \mathbb{R}$. We let denote by $\mathcal{L}_H$ the sub-Riemannian (a.k.a. Carnot-Carathéodory) distance

1. If $K \geq 0$, then the metric measure space $(M, d, \mu)$ satisfies the volume doubling property and supports a 2-Poincaré inequality, i.e. there exist constants $C_P, C_V > 0$, depending only on $K$, $m$, $n$, for which one has for every $p \in M$ and every $\epsilon > 0$:
\[ p(B(p, \epsilon)) \leq C_P \mu(B(p, \epsilon)). \tag{6} \]

2. If $K > 0$, then $M$ is compact with a finite fundamental group and $\text{diam}(M, d) \leq 2 \sqrt{\frac{m+n}{n-m}} \frac{\mu(B)}{\mu(M)} \tag{7} \]

3. If $\text{K} > 0$, then the first non-zero eigenvalue of the sub-Laplacian $-\Delta_H$ satisfies
\[ \lambda_1 \geq \frac{\pi K}{n-m} \tag{8} \]

References


