H-type foliations

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Abstract

With a view toward sub-Riemannian geometry, we introduce and study H-type foliations. These structures are natural generalizations of K-contact geometries which encompass as special cases K-contact manifolds, twistor spaces, 3K contact manifolds and H-type groups. Under an horizontal Ricci curvature lower bound, we prove on those structures sub-Riemannian diameter upper bounds and first eigenvalue estimates for the sub-Laplacian. Then, using a result by Moroianu-Semmelmann, we classify the H-type foliations that carry a parallel horizontal Clifford structure. Finally, we prove an horizontal Einstein property and compute the horizontal Ricci curvature of those spaces in codimension more than 2.

Background

A sub-Riemannian manifold is a smooth manifold $\mathbb{M}$ equipped with a bracket generating distribution $\mathcal{H} \subset T\mathbb{M}$ and a fiber inner product $g_{\mathcal{H}}$ on $\mathcal{H}$. We require the existence of a transverse totally geodesic and integrable complement $\mathcal{V}$. From this, we introduce and study a new class of sub-Riemannian manifolds generalizing the H-type groups introduced by Kaplan in [5]. We call such manifolds H-type sub-Riemannian manifolds.

Proposition 1 (Hladky [4])**.** *There exists a unique metric connection ∇ on $\mathbb{M}$, called the Bott connection of the foliation, such that:*

1. $\mathcal{H}$ and $\mathcal{V}$ are ∇-parallel, i.e. for every $X \in \Gamma(\mathcal{H})$, $Y \in \Gamma(T\mathbb{M})$ and $Z \in \Gamma(\mathcal{V})$,

$$\nabla_Y X \in \Gamma(\mathcal{H}), \quad \nabla_Y Z \in \Gamma(\mathcal{V}); \tag{1}$$

2. The torsion T of ∇ satisfies

$$T(\mathcal{H},\mathcal{H}) \subset \mathcal{V}, \qquad T(\mathcal{H},\mathcal{V}) = 0, \qquad T(\mathcal{V},\mathcal{V}) = 0. \tag{2}$$

This connection is better suited to the study of the foliation structure, as it preserves the horizontal and vertical bundles.

Claim 2 (Kaplan J map)**.**
For $Z \in \Gamma(\mathcal{V})$ there exists a unique skew-symmetric fiber endomorphism $J_Z \colon \Gamma(\mathcal{H}) \to \Gamma(\mathcal{H})$ such that

$$g_{\mathcal{H}}(J_Z X, Y) = g_{\mathcal{V}}(Z, T(X,Y)) \tag{3}$$

for every $X, Y \in \Gamma(\mathcal{H})$, $Z \in \Gamma(\mathcal{V})$, where $T(X,Y) = \nabla_X Y - \nabla_Y X$ is the torsion tensor of the Bott connection.

Definition 3 (H-type sub-Riemmanian Manifold)**.**
Let $\mathbb{M}$ be a smooth, oriented, connected, manifold with dimension $n+m$, equipped with a Riemannian foliation with bundle-like complete metric g and totally geodesic m-dimensional leaves.
If

$$\langle J_Z X, J_Z Y \rangle = \|Z\|^2 \langle X, Y \rangle \tag{4}$$

we say that $(\mathbb{M}, \mathcal{H}, g)$ is an H-type foliation.

Due to their symmetries, H-type sub-Riemannian manifolds provide an ideal framework to develop a program reducing the study of global geometric, metric, or analytic properties of the ambient space to the study of local sub-Riemannian curvature type invariants.

Main Results

Yang-Mills Property

We show that all H-type foliations are Yang-Mills; as a consequence, the sub-Laplacian of an H-type foliation satisfies a simple Bochner's type formula and the validity of the generalized curvature dimension inequality is only controlled by the horizontal Ricci curvature of the Bott connection.

Proposition 4. *Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation such that $\mathbf{Ric}_{\mathcal{H}} \geq K g_{\mathcal{H}}$ with $K \in \mathbb{R}$. Then $(\mathbb{M}, \mathcal{H}, g)$ satisfies the generalized curvature dimension inequality $CD\big(K, \frac{n}{4}, m, n\big)$, i.e. for every $f \in C^\infty(\mathbb{M})$ and $\varepsilon > 0$, one has the following Bochner's type inequality:*

$$\frac{1}{2}\left(\Delta_{\mathcal{H}}\|\nabla_{\mathcal{H}} f\|^2 - 2\langle \nabla_{\mathcal{H}} f, \nabla_{\mathcal{H}} \Delta_{\mathcal{H}} f\rangle\right) + \frac{\varepsilon}{2}\left(\Delta_{\mathcal{H}}\|\nabla_{\mathcal{V}} f\|^2 - 2\langle \nabla_{\mathcal{V}} f, \nabla_{\mathcal{V}} \Delta_{\mathcal{H}} f\rangle\right) \geq \frac{1}{n}(\Delta_{\mathcal{H}} f)^2 + \left(K - \frac{m}{\varepsilon}\right)\|\nabla_{\mathcal{H}} f\|^2 + \frac{n}{4}\|\nabla_{\mathcal{V}} f\|^2. \tag{5}$$

The consequences of the generalized curvature dimension inequality have been extensively studied recently (see [1, 2, 3]), in our setting we will have

Corollary 5. *Let $(\mathbb{M}, \mathcal{H}, g)$ be a complete H-type foliation with $\mathbf{Ric}_{\mathcal{H}} \geq K g_{\mathcal{H}}$ with $K \in \mathbb{R}$. Let us denote by d the sub-Riemannian (a.k.a. Carnot-Carathéodory) distance.*

1. If $K \geq 0$, then the metric measure space $(\mathbb{M}, d, \mu)$ satisfies the volume doubling property and supports a 2-Poincaré inequality, i.e. there exist constants $C_D, C_P > 0$, depending only on K, n, m, for which one has for every $p \in \mathbb{M}$ and every $r > 0$:

$$\mu(B(p, 2r)) \leq C_D\, \mu(B(p, r)), \tag{6}$$

$$\int_{B(p,r)} |f - f_B|^2 d\mu_g \leq C_P r^2 \int_{B(p,r)} \|\nabla_{\mathcal{H}} f\|^2 d\mu_g, \tag{7}$$

for every $f \in C^1(B(p,r))$, where we have let $f_B = \mu_g(B)^{-1}\int_B f d\mu_g$, with $B = B(p,r)$.

2. If $K > 0$, then $\mathbb{M}$ is compact with a finite fundamental group and

$$\mathbf{diam}(\mathbb{M}, d) \leq 2\sqrt{3}\pi\sqrt{\frac{(n+4m)(n+6m)}{nK}}. \tag{8}$$

3. If $K > 0$, then the first non zero eigenvalue of the sub-Laplacian $-\Delta_{\mathcal{H}}$ satisfies

$$\lambda_1 \geq \frac{nK}{n+3m-1}. \tag{9}$$

Classification of H-type foliations

We define the notion of parallel Clifford structure; H-type foliations with a parallel horizontal Clifford structure are to general H-type foliations what Sasakian and 3-Sasakian manifolds are respectively to K-contact and 3K-contact manifolds.

Definition 6.
Let $(\mathbb{M}, g, \mathcal{H})$ be an H-type foliation with horizontally parallel torsion, that is

$$\nabla_{\mathcal{H}} J = 0. \tag{10}$$

We say that $(\mathbb{M}, g, \mathcal{H})$ is an H-type foliation with a parallel horizontal Clifford structure if there exists a smooth bundle map $\Psi : \mathcal{V} \times \mathcal{V} \to \mathbf{Cl}_2(\mathcal{V})$ such that for every $Z_1, Z_2 \in \Gamma(\mathcal{V})$

$$(\nabla_{Z_1} J)_{Z_2} = J_{\Psi(Z_1, Z_2)}. \tag{11}$$

When it exists, the Ψ map has a rigid structure:

Theorem 7. *Let $(\mathbb{M}, g, \mathcal{H})$ be an H-type foliation with parallel horizontal Clifford structure. Then, there exists a constant $\kappa \in \mathbb{R}$ such that for every $u, v \in \mathcal{V}$*

$$\Psi(u, v) = -\kappa(u \cdot v + \langle u, v\rangle). \tag{12}$$

Moreover the sectional curvature of the leaves of the foliation associated to $\mathcal{V}$ is constant equal to κ^2. In particular, if the torsion is completely parallel, the leaves are flat.

This condition is analogous to the notion of Clifford structure introduced by A. Moroianu and U. Semmelman [6]. With it, we can arrive at the notion of a horizontal Einstein property:

Theorem 8. *Let $(\mathbb{M}, g, \mathcal{H})$ be an H-type foliation with a parallel horizontal Clifford structure with $m \geq 2$. Then*

- *If $m \neq 3$, $\mathbf{Ric}_{\mathcal{H}} = \kappa\left(\frac{n}{4} + 2(m-1)\right) g_{\mathcal{H}}$.*
- *If $m = 3$ and $(\mathbb{M}, g, \mathcal{H})$ is of quaternionic type then $\mathbf{Ric}_{\mathcal{H}} = \kappa\left(\frac{n}{2} + 4\right) g_{\mathcal{H}}$.*
- *If $m = 3$ and $(\mathbb{M}, g, \mathcal{H})$ is not of quaternionic type, then at any point, $\mathcal{H}$ orthogonally splits as a direct sum $\mathcal{H}^+ \oplus \mathcal{H}^-$ and for $X, Y \in \Gamma(\mathcal{H})$,*

$$\mathbf{Ric}_{\mathcal{H}}(X, Y) = \kappa\left(\frac{n}{4} + 4\right)\langle X, Y\rangle + \frac{\kappa}{4}(\dim \mathcal{H}^+ - \dim \mathcal{H}^-)\langle \sigma(X), Y\rangle, \tag{13}$$

where $\sigma = \mathbf{Id}_{\mathcal{H}^+} \oplus (-\mathbf{Id}_{\mathcal{H}^-})$.

This gives the following table classifying H-type foliations with parallel horizontal Clifford structure coming from a globally defined submersion $\pi \colon \mathbb{M} \to \mathbf{B}$ with $\kappa > 0$ (an analogous table exists for $\kappa < 0$).

$\mathbb{M}$	$\mathbb{B}$	Fiber	$\mathbf{rank}(\mathcal{H})$	$\mathbf{rank}(\mathcal{V})$
Twistor space	Quaternion-Kähler with positive scalar curvature	$\mathbb{S}^2$	$4k$	2
3-Sasakian	Quaternion-Kähler with positive scalar curvature	$\mathbb{S}^3$	$4k$	3
Quaternion-Sasakian	Product of two quaternion-Kähler with positive scalar curvature	$\mathbb{R}P^3$	$4k$	3
$\frac{\mathbf{Sp}(q^+{+}1)\times\mathbf{Sp}(q^-{+}1)}{\mathbf{Sp}(q^+)\times\mathbf{Sp}(q^-)\times\mathbf{Sp}(1)}$	$\mathbb{H}P^{q^+} \times \mathbb{H}P^{q^-}$	$\mathbb{S}^3$	$4(q^+ + q^-)$	3
$\frac{\mathbf{Sp}(k+2)}{\mathbf{Sp}(k)\times\mathbf{Spin}(4)}$	$\frac{\mathbf{Sp}(k+2)}{\mathbf{Sp}(k)\times\mathbf{Sp}(2)}$	$\mathbb{S}^4$	$8k$	4
$\frac{\mathbf{SU}(k+4)}{\mathbf{S}(\mathbf{U}(k)\times\mathbf{Sp}(2)\mathbf{U}(1))}$	$\frac{\mathbf{SU}(k+4)}{\mathbf{S}(\mathbf{U}(k)\times\mathbf{U}(4))}$	$\mathbb{R}P^5$	$8k$	5
$\frac{\mathbf{SO}(k+8)}{\mathbf{SO}(k)\times\mathbf{Spin}(7)}$	$\frac{\mathbf{SO}(k+8)}{\mathbf{SO}(k)\times\mathbf{SO}(8)}$	$\mathbb{R}P^7$	$8k$, $k \geq 3$, k odd	7
$\frac{\mathbf{Spin}(k+8)}{\mathbf{SO}(k)\times\mathbf{Spin}(7)}$	$\frac{\mathbf{SO}(k+8)}{\mathbf{SO}(k)\times\mathbf{SO}(8)}$	$\mathbb{S}^7$	$8k$, $k = 1$, k even	7
Exceptional cases				
$\frac{F_4}{\mathbf{Spin}(8)}$	$\frac{F_4}{\mathbf{Spin}(9)} = \mathbb{O}P^2$	$\mathbb{S}^8$	16	8
$\frac{E_6}{\mathbf{Spin}(8)\mathbf{U}(1)}$	$\frac{E_6}{\mathbf{Spin}(10)\mathbf{U}(1)} = (\mathbb{C}\otimes\mathbb{O})P^2$	$\mathbb{S}^9$	32	9
$\frac{E_7}{\mathbf{Spin}(11)\mathbf{SU}(2)}$	$\frac{E_7}{\mathbf{Spin}(12)\mathbf{SU}(2)} = (\mathbb{H}\otimes\mathbb{O})P^2$	$\mathbb{S}^{11}$	64	11
$\frac{E_8}{\mathbf{Spin}(15)}$	$\frac{E_8}{\mathbf{Spin}^+(16)} = (\mathbb{O}\otimes\mathbb{O})P^2$	$\mathbb{S}^{15}$	128	15

Table 1: H-type submersions with a parallel horizontal Clifford structure and $\kappa > 0$.

Sub-Riemannian diameters and first eigenvalue estimates

Combining the consequences of the Yang-Mills and horizontal Einstein properties, we are able to conclude

Corollary 9. *Let $(\mathbb{M}, \mathcal{H}, g)$ be a complete H-type foliation with a parallel horizontal Clifford structure: $\Psi(Z, W) = -\kappa(Z \cdot W + \langle Z, W\rangle)$, with $\kappa > 0$. Then, $\mathbb{M}$ is compact with finite fundamental group. Moreover, denoting the sub-Riemannian diameter $\mathbf{diam}(\mathbb{M}, d)$ and the first eigenvalue of the sub-Laplacian by λ_1 we have the following estimates:*

- *If $m \neq 3$,*

$$\mathbf{diam}(\mathbb{M}, d) \leq 4\sqrt{3}\frac{\pi}{\sqrt{\kappa}}\sqrt{\frac{(n+4m)(n+6m)}{n(n+8(m-1))}}, \qquad \lambda_1 \geq \frac{\kappa n(n+8(m-1))}{4 \quad n+3m-1}. \tag{14}$$

- *If $m = 3$ and $(\mathbb{M}, \mathcal{H}, g)$ is of quaternionic type, then*

$$\mathbf{diam}(\mathbb{M}, d) \leq 2\sqrt{6}\frac{\pi}{\sqrt{\kappa}}\sqrt{\frac{(n+12)(n+18)}{n(n+8)}}, \qquad \lambda_1 \geq \frac{n\kappa}{2}. \tag{15}$$

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