Green and Stokes' Theorems

Overview

Green and Stokes' Theorems are generalizations of the Fundamental Theorem of Calculus, letting us relate double integrals over 2 dimensional regions to single integrals over their boundary; as you study this section, it's very important to try to keep this idea in mind. They will allow us to compute many integrals that arise in real life situations, and give us a much deeper understanding of the relationship between multivariate forms of the derivative and integrals.

13.4 Green's Theorem

Begin by recalling the Fundamental Theorem of Calculus:

$$\int_a^b f'(x)\ dx = f(b) - f(a)$$

and the more recent Fundamental Theorem for Line Integrals for a curve C parameterized by $\vec{r}(t)$ with $a \leq t \leq b$

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

which amounts to saying that if you're integrating the derivative a function (in some sense) along an interval or a curve, you can instead just sum something computed at each end. Is this always possible? No, only if we can find the antiderivative of f, or the potential function (depending on the case.)

Our hope in this section is to find an equivalent for double integrals. That is:

$$\iint_R f(x,y)\ dA = \int_{\partial R} g(x,y)\ d\vec{r}$$

where R is a region, and ∂R is its boundary (which is a curve) parameterized by $\vec{r}$. Stress that ∂R must be oriented counterclockwise, for reasons related to complex numbers. Definitely draw a picture.

The problem: What does it mean for f to be the derivative of g here? Green's Theorem provides an answer:

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\partial R} \langle P, Q\rangle \cdot d\vec{r}$$

Hopefully this convincingly looks like some sort of derivative. A weird one, but still. Make sure to notice the fact that you take the derivative of the second component (Q) wrt. the first variable (x) and vice versa, just like in the test for conservative vector fields. In fact, if $\langle P, Q\rangle$ is conservative, what does that mean for this integral? (It's 0, since we're integrating around a closed curve.)

It's interesting know that the minus sign that appears in Green's Theorem is actually a consequence of complex numbers; I'll elaborate more on this when I cover the Divergence Theorem.

Exmaples

1. Compute directly, then using Green's Theorem

$$\iint_R (2x - 3y)\, dA$$

where R is a disk of radius 2 centered at the origin. HINT: Use polar coordinates.

<u>ANSWER:</u> Directly,

$$\begin{aligned}
\iint_R (2x - 3y)\ dA &= \int_0^{2\pi} \int_0^2 2r\cos(\theta) - 3r\sin(\theta))r\ drd\theta \\
&= \int_0^{2\pi} \int_0^2 r^2(2\cos(\theta) - 3\sin(\theta))\ drd\theta \\
&= \int_0^{2\pi} (2\cos(\theta) - 3\sin(\theta))\ d\theta \int_0^2 r^2\ dr \\
&= (-2\sin(\theta) - 3\cos(\theta)|_{\theta=0}^{\theta=2\pi} \left(\frac{1}{3}r^3\right|_{r=0}^{r=2} \\
&= 0
\end{aligned}$$

Using Green's Theorem, we parameterize the circle as $\vec{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle$, $0 \le t \le 2\pi$ and then

$$\begin{aligned}
\iint_R (2x - 3y)\ dA &= \int_C \left\langle \frac{3}{2}y^2, x^2 \right\rangle \cdot d\vec{r} \\
&= \int_0^{2\pi} \left\langle \frac{3}{2}(2\sin(t))^2, (2\cos(t))^2 \right\rangle \cdot \langle 2\cos(t), 3\sin(t)\rangle\ dt \\
&= \int_0^{2\pi} (12\sin(t)\cos^2(t) + 12\cos(t)\sin^2(t))\ dt \\
&= \int_1^1 -12u^2\ du + \int_0^0 12v^2\ dv \\
&= 0
\end{aligned}$$

where I made the substitutions $u = \cos(t), v = \sin(t)$.

2. Compute, using Green's Theorem:

$$\int_C \langle 3y - e^{\sin(x)}, 7x + \sqrt{y^4 + 1}\rangle \cdot d\vec{r}$$

where C is the boundary of the rectangle oriented counterclockwise with vertices (0,0), (0,3), (2,3), (2,0). Can this be done directly?

ANSWER: Using Green's Theorem,

$$\begin{aligned}
\int_C \langle 3y - e^{\sin(x)}, 7x + \sqrt{y^4+1}\rangle \cdot d\vec{r} &= \iint_R \frac{\partial}{\partial x}(7x + \sqrt{y^4+1}) - \frac{\partial}{\partial y}(3y - e^{\sin(x)})\ dA \\
&= \int_0^3 \int_0^2 (7-3)\ dxdy \\
&= 4 \cdot 3 \cdot 2 \\
&= 24
\end{aligned}$$

and no, it's impossible to compute this in closed form (aka without Taylor series) unless you use Green's Theorem.

13.6 Surface Integrals

Okay, this section is essentially just a new type of integration. Recall that we were able to generalize integrals along curves by defining

$$\int_C f(x,y)\ d\vec{r} = \int_a^b f(x(t),y(t))\ \|\vec{r}\,'(t)\|\ dt$$

and

$$\int_C \vec{F}(x,y) \cdot d\vec{r} = \int_a^b \vec{F}(x(t),y(t)) \cdot \vec{r}\,'(t)\ dt$$

where C is a curve parameterized as $\vec{r}(t), a \leq t \leq b$. We would like to do the same for surfaces.

Since surfaces are two dimensional, it makes sense that this should be a double integral. Remember how we could write the vector equation of lines using a vector parallel to the line, but in order to write the vector equation of a plane we needed a vector orthogonal to the plane? The same happens here. That is, at every point of a curve $\vec{r}(t)$ its velocity $\vec{r}\,'(t)$ is parallel to the curve (it actually defines the tangent line), so it shows up in the formula for line integrals.

For surface integrals, we need a normal vector $\vec{n}$. Recall from the formula for tangent planes to parameterized surfaces $\vec{r}(u,v)$ we can compute $\vec{n} = \vec{r}_u \times \vec{r}_v$. This is why we define the surface integral over a parameterized surface

$$S : \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle.$$

with $a \leq u \leq b, c \leq v \leq d$ as:

$$\iint_S f(x,y,z)\ d\vec{S} = \int_c^d \int_a^b f(x(u,v),y(u,v),z(u,v))\ \|\vec{r}_u(u,v) \times \vec{r}_v(u,v)\|\ dudv$$

and similarly

$$\iint_S \vec{F}(x,y,z) \cdot d\vec{S} = \int_c^d \int_a^b \vec{F}(x(u,v),y(u,v),z(u,v)) \cdot (\vec{r}_u(u,v) \times \vec{r}_v(u,v))\ dudv$$

this is a little overwhelming, but dropping all of the independent variables and writing $\vec{n} = \vec{r}_u \times \vec{r}_v$ it really just says that

$$\iint_S f d\vec{S} = \int_c^d \int_a^b f\|\vec{n}\| \; dudv$$

and

$$\iint_S \vec{F} \cdot d\vec{S} = \int_c^d \int_a^b \vec{F} \cdot \vec{n} \; dudv$$

Make sure you see why this formula is really the same idea as for line integrals. Importantly, if a surface is closed (that is, it encloses completely a 3-dimensional volume) we say the surface is positively-oriented if $\vec{n}$ is pointing outwards from the volume. For open surfaces (surfaces that aren't closed) this idea isn't well defined, so we don't bother. If you need a positive orientation and you have a negatively-oriented one, just use the negative of the normal vector that you already have.

To interpret what surface integrals actually measure, understand that a surface integral will be very positive if the vector field $\vec{F}$ points in the same direction as the normal vectors; that is it measures how much $\vec{F}$ passes through the surface. For a closed surface with positive orientation, the surface integral measures how much the field is leaving the enclosed volume.

One other word of caution: there exist surfaces which are unorientable: that is, it's impossible to choose a normal vector. This happens when the surface has "only one side". This may seem nonsensical, but there's an easy example: the Möbius strip. The issue is that if you try to pick a normal vector at any point, you can smoothly move this vector around the surface all the way around until you arrive at exactly the opposite vector. There's no way around this, which means that surface integrals do not make sense on this surface and cannot be computed.

Examples

1. Let S be the parallelogram with parameteric equations

$$\begin{cases} x = u + v \\ y = u - v \\ z = 1 + 2u \end{cases} \qquad 0 \leq u \leq 2, 0 \leq v \leq 1$$

Compute

$$\iint_S (x + y + z) \; dS$$

ANSWER: We begin by rewriting the parameteric equations in vector form, and computing the magnitude of the normal vector (choice of orientation is irrelevant, because magnitude is always positive).

$$\vec{r}(u, v) = \langle u + v, u - v, 1 + 2u\rangle, \quad 0 \leq u \leq 2, 0 \leq v \leq 1$$

so that

$$\vec{n} = \vec{r}_u \times \vec{r}_v = \langle 1, 1, 2\rangle \times \langle 1, -1, 0\rangle = \langle 2, 2, -2\rangle$$

and then

$$dS = \|\vec{n}\|\ dudv = 2\sqrt{3}\ dudv$$

Finally,

$$\begin{aligned}\iint_S (x+y+z)\ dS &= \int_0^1 \int_0^2 ((u+v)+(u-v)+(1+2u))(2\sqrt{3})\ dudv \\ &= 2\sqrt{3}\int_0^1 \int_0^2 (1+4u)\ dudv \\ &= 2\sqrt{3}\int_0^1 \left(u+2u^2\right|_{u=0}^{u=2}\ dv \\ &= 2\sqrt{3}\int_0^1 10\ dv \\ &= 20\sqrt{3}\end{aligned}$$

2. Let S be the surface given by the graph of $z = xy, 0 \le x \le 1, 0 \le y \le 1$. Oriented with $\vec{n}$ upwards (towards positive z). Compute

$$\iint_S \langle 1, y, x\rangle \cdot d\vec{S}$$

ANSWER: We must parameterize the surface. Since it is a graph, we can make the simple substitutions $u = x, v = y$ and parameterize it as

$$\vec{r}(u,v) = \langle u, v, uv\rangle, \quad 0 \le u \le 1, 0 \le v \le 1$$

We compute the normal vector

$$\vec{n} = \vec{r}_u \times \vec{r}_v = \langle 1, 0, v\rangle \times \langle 0, 1, u\rangle = \langle -v, -u, 1\rangle$$

Since this $\vec{n}$ has positive z component we are done, if it didn't we could multiply it by negative 1. Then the integral becomes

$$\begin{aligned}\iint_S \langle 1, y, x\rangle \cdot d\vec{S} &= \int_0^1 \int_0^1 \langle 1, v, u\rangle \cdot \langle -v, -u, 1\rangle\ dudv \\ &= \int_0^1 \int_0^1 (-v - uv + u)\ dudv \\ &= \int_0^1 \left(-uv - \frac{1}{2}u^2 v + \frac{1}{2}u^2\right|_{u=0}^{u=1}\ dv \\ &= \int_0^1 \left(-\frac{3}{2}v + \frac{1}{2}\right)\ dv \\ &= \left(-\frac{3}{4}v^2 + \frac{1}{2}v\right|_{v=0}^{v=1} \\ &= -\frac{1}{2}\end{aligned}$$

13.7 Stokes' Theorem

Now that we have surface integrals, we can talk about a much more powerful generalization of the Fundamental Theorem: Stokes' Theorem. Green's Theorem let us take an integral over a 2-dimensional region in $\mathbb{R}^2$ and integrate it instead along the boundary; Stokes' Theorem allows us to do the same thing, but for surfaces in $\mathbb{R}^3$! Here's the statement:

$$\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$$

where ∂R is the boundary of R oriented in the same direction as the surface; that is, if you use the right-hand rule with thumb and fingers, your thumb will point in the direction of the normal vector for the surface and your fingers will point in the same direction as the velocity vector for the curve.

To interpret this a bit, remember that we keep thinking intuitively that the integral of the "derivative" of a function or vector field over some region is equal the integral of that vector field over the boundary of the region. And we learned last class that curl is a way of differentiating vector fields, so this formula does make some sense.

Hopefully you can see a superficial resemblence to Green's Theorem. It turns out, this actually contains Green's Theorem! Here's the trick: imagine the plane $\mathbb{R}^2$ in Green's Theorem is actually the xy-plane in $\mathbb{R}^3$, and choose its normal vector $\vec{n}$ to be the unit vector in the z-direction. That is, $\vec{n} = \hat{k}$. Importantly, your vector field $\vec{F} = \langle P, Q \rangle$ has to be rewritten as a vector field in $\mathbb{R}^3$, so choose it to be the vector field with z-component 0; that is, let $\vec{F} = \langle P, Q, 0 \rangle$. Now, if we try to compute the integral in Green's Theorem but use Stoke's Theorem, we get:

$$\begin{aligned}
\int_{\partial R} \vec{F} \cdot d\vec{r} &= \iint_S \text{curl}(\langle P, Q, 0 \rangle) \cdot d\vec{S} \\
&= \iint_R \left\langle -\frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle \cdot \hat{k} \; dudv \\
&= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \; dA
\end{aligned}$$

which is exactly what Green's Theorem says!! In fact, it should make you feel a bit better about Green's Theorem since the term $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ seemed to come out of nowhere, and now we can see that this is actually coming from the curl of $\vec{F}$ in $\mathbb{R}^3$!

Another comment: we already know that if $\vec{F}$ is conservative then its integral around a closed curve is 0; that's because the Fundamental Theorem for Line Integrals says:

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) = 0$$

since $\vec{r}(a) = \vec{r}(b)$ for closed curves. Pretend for a moment that we didn't already know this, and try to apply Stokes' Theorem:

$$\int_C \nabla f \cdot d\vec{r} = \iint_S \text{curl}(\nabla f) \cdot d\vec{S}$$

for any surface S bounded by C. This is already an interesting point; you can pick any surface that has C as its boundary, which is a really large collection to pick from. More than that, however, let's compute $\text{curl}(\nabla f)$:

$$\begin{aligned}\text{curl}(\nabla f) &= \text{curl}(\langle f_x, f_y, f_z\rangle) \\ &= \langle f_{yz} - f_{zy}, f_{xz} - f_{zx}, f_{yx} - f_{xy}\rangle \\ &= \langle 0, 0, 0\rangle\end{aligned}$$

using Clairaut's Theorem! So

$$\int_C \nabla f \cdot d\vec{r} = \iint_S \langle 0, 0, 0\rangle \cdot d\vec{S} = 0$$

and we've found another proof of the fact that the integral around a closed curve of a conservative vector field is 0.

I want to comment here; this isn't really the full force of Stokes' Theorem. Notice that the original Fundamental Theorem of Caclculus and the Fundamental Theorem of Line Integrals related a 1-dimensional integral to a 0-dimensional integral (a sum), while Green and Stokes' Theorems relate 2-dimensional integrals to 1-dimensional integrals. We'll soon see the Divergence Theorem, which relates a 3-dimensional integral to 2-dimensional integrals, but this isn't really the end of the story: it's essentially enough for physics/engineering/most other applications since we live in 3-dimensional space, but in fact there is a very general result relating n-dimensional integrals to (n-1)-dimensional integrals that contains all of the results we've seen.

Okay, examples that you can work or they can (some combo is best):

Examples

1. Let S_1 be the surface $z = x^2 + y^2 - 4, \sqrt{x^2 + y^2} \leq 2$ and let S_2 be the surface $\sqrt{x^2 + y^2 + z^2} = 2$, both oriented with normal vectors pointing upwards (with positive z-component). Let $\vec{F}$ be any vector field. What can you say about the relationship between

   $$\iint_{S_1} \text{curl}(\vec{F}) \cdot d\vec{S_1}$$

   and

   $$\iint_{S_2} \text{curl}(\vec{F}) \cdot d\vec{S_2}$$

   ANSWER: They are equal, since by Stokes' Theorem they are both equal to

   $$\int_C \vec{F} \cdot d\vec{r}$$

   where C is the circle $x^2 + y^2 = 4, z = 0$ with the same orientation.

2. Compute

$$\iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$$

Where S is the surface $\sqrt{x^2 + y^2 + z^2} = 2$ oriented in the positive z-direction and $\vec{F} = \langle -3y, 3x, xy^3 \rangle$.

<u>ANSWER:</u> From the previous problem, we already know that

$$\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$$

where C is the circle $x^2 + y^2 = 4, z = 0$ with counterclockwise orientation. We parameterize the circle as

$$\vec{r}(t) = \langle 2\cos(t), 2\sin(t), 0 \rangle, 0 \le t \le 2\pi$$

and so

$$d\vec{r} = \vec{r}\,'(t)dt = \langle -2\sin(t), 2\cos(t), 0 \rangle$$

then

$$\begin{aligned}
\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} &= \int_0^{2\pi} \langle -6\sin(t), 6\cos(t), 16\sin(t)\cos^3(t) \rangle \cdot \langle -2\sin(t), 2\cos(t), 0 \rangle \; dt \\
&= \int_0^{2\pi} (12\sin^2(t) + 12\cos^2(t)) \; dt \\
&= \int_0^{2\pi} 12 \; dt \\
&= 24\pi
\end{aligned}$$