Holonomy of Riemannian Manifolds and Berger's Classification

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- Smooth structure
- Riemannian metric

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Definition

If there is a collection Φ of smooth (infinitely differentiable) transition maps that cover $\mathbb{M}$, we call $(\mathbb{M}, \mathcal{T}, \Phi)$ a smooth manifold.

The Tangent Bundle

- Associated to every point $p \in \mathbb{M}$ we have a copy of $\mathbb{R}^n$. We call these tangent spaces $T_p\mathbb{M}$, and their disjoint union the tangent bundle $T\mathbb{M} = \cup_{p\in\mathbb{M}} T_p\mathbb{M}$.

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- The vectors in $T_p\mathbb{M}$ can be put into 1-1 correspondence with the smooth curves $\gamma\colon [0,1] \to \mathbb{M}$ passing through p "in the same direction", so we view elements of $T_p\mathbb{M}$ as tangent vectors to curves.

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- If at every point p in $\mathbb{M}$ we choose a tangent vector $X_p \in T_p\mathbb{M}$, we call the collection a vector field $X \in \mathfrak{X}(\mathbb{M})$.

Riemannian Manifolds

- In order to make sense of length on the tangent spaces, we let

$$g_p(\cdot,\cdot)\colon\ T_p\mathbb{M}\times T_p\mathbb{M}\to\mathbb{R}$$

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- If these inner products vary smoothly as a function of p, we call it a <u>Riemannian metric</u> denoted g.
- A smooth manifold with a Riemannian metric $(\mathbb{M},\mathcal{T},\Phi,g)$ is called a <u>Riemannian manifold</u>.

Tangent Plane Identification Problem

Suppose we have two points $p, q \in \mathbb{M}$, and a vector $X_p \in T_p\mathbb{M}$. By construction,

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This leaves the question: what vector in $T_q\mathbb{M}$ should correspond to X_p?

We can visualize connecting p and p by a curve, and "transporting" X_p along the curve to q, but it is not clear how to do this.

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Any such ∇ is called a connection.

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Given a curve γ from p to q and a γ-parallel vector field X, we say that X_q is the parallel transport of X_p along γ, which we denote

$$\begin{aligned} \tau_\gamma \colon T_p\mathbb{M} &\to T_q\mathbb{M} \\ X_p &\mapsto \tau_\gamma X_p = X_q \end{aligned}$$

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Theorem (Fundamental Theorem of Riemannian Geometry)

There exists a unique connection ∇ *such that*

1. $\nabla_X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$
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This connection is called the Levi-Civita connection.

Riemann Curvature Tensor

It will be important later to discuss the idea of curvature, which is quantified by the Riemann curvature tensor

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

This is a tensor (not obvious) and measures the noncommutativity of the connection ∇.

Tangent Space Identification?

However, this process still has not answered the problem of identifying the tangent vectors: for each curve from p to q we have an identification of $T_p\mathbb{M}$ with $T_q\mathbb{M}$.

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While this has no simple resolution, it introduces an interesting question. What are all possible identifications of the tangent spaces?

Holonomy

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Definition

Fix a point $p \in \mathbb{M}$ and consider all possible curves γ in $\mathbb{M}$ that both start and end at p. Associated to each one there is an automorphism of $T_p\mathbb{M}$. We call the collection of automorphisms the Riemannian holonomy group of $\mathbb{M}$ at p, denoted $\mathrm{Hol}(\mathbb{M}, p)$.

Immediate Properties of the Holonomy Group

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- If γ is a path connecting p to q, we see that

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  and so the holonomy group $\mathrm{Hol}(\mathbb{M})$ is independent of the basepoint.
- Since parallel transport is a linear isometry, it is clear that

$$\mathrm{Hol}(\mathbb{M}) \subseteq O(n)$$

Restricted Holonomy and the Fundamental Group

Definition

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3. ∇ is flat if and only if $\mathrm{Hol}^0(\mathbb{M})$ is trivial.

Example: $\mathbb{R}^2 \times S^1$

Consider $\mathbb{M} = \mathbb{R}^2 \times S^1$, which can be constructed as

$$\mathbb{R}^2 \times S^1 \cong \mathbb{R}^2 \times [0,1]/\sim$$

where $((x,y),0) \sim (f(x,y),1)$ for some isomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$.

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- If f_α is the linear rotation of $\mathbb{R}^2$ by angle α, we have a flat Riemannian manifold $(\mathbb{M}, g_\alpha)$. It is complete, noncompact, and $\mathrm{Hol}^0(\mathbb{M}) = 0$.

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- If f_α is the linear rotation of $\mathbb{R}^2$ by angle α, we have a flat Riemannian manifold $(\mathbb{M}, g_\alpha)$. It is complete, noncompact, and $\mathrm{Hol}^0(\mathbb{M}) = 0$.
- However, $\mathrm{Hol}(\mathbb{M}) \subseteq O(2)$ is the group generated by f_α; finite if $\alpha = k\pi, k \in \mathbb{Q}$, and isomorphic to $\mathbb{Z}$ otherwise.

Orientability and $SO(n)$

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- It follows that if $(\mathbb{M}, g)$ is orientable, it must hold that

$$\mathrm{Hol}(\mathbb{M}) \subseteq SO(n)$$

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- We can then define an integrable almost complex structure J by
$$g(X, Y) = \omega(JX, Y)$$
- By the Newlander-Nirenburg Theorem, $(\mathbb{M}, g)$ is Kähler.

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3. *There exists $p \in \mathbb{M}$ and on $T_p^{(r,s)}\mathbb{M}$ a tensor α invariant by $\mathrm{Hol}(\mathbb{M})$.*

Calabi-Yau manifolds and $SU(n)$

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- Such manifolds are called Calabi-Yau, and are necessarily Ricci-flat.

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- Again applying the fundamental principle,

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- Equivalently, at every point $(p_1, p_2) \in \mathbb{M}_1 \times \mathbb{M}_2$, the tangent plane splits as the Euclidean product:

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- A Riemannian manifold (locally) isometric to a Riemannian product is called (locally) reducible.

deRham Decomposition

Theorem (deRham)

Let $(\mathbb{M}, g)$ be a complete, simply connected Riemannian manifold. Then $\mathrm{Hol}(\mathbb{M})$ is reducible as a direct product if and only if $(\mathbb{M}, g)$ is reducible as a Riemannian product.

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That is, the splitting

$$(\mathbb{M}, g) = (\mathbb{M}_1 \times \mathbb{M}_2, g_1 \times g_2)$$

is equivalent to the splitting

$$\mathrm{Hol}(\mathbb{M}, g) = \mathrm{Hol}(\mathbb{M}_1, g_1) \times \mathrm{Hol}(\mathbb{M}_2, g_2)$$

Lassos and Ambrose-Singer

By considering “lassos”, that is small loops conjugated by parallel transport, we have the following:

Theorem (Ambrose-Singer)

The Lie algebra $\mathfrak{hol}(p)$ *of* $\mathrm{Hol}(\mathbb{M})$ *at* $p \in \mathbb{M}$ *is the subalgebra of* $\mathfrak{so}(T_p\mathbb{M})$ *generated by the elements*

$$\tau_\gamma^{-1} \circ R(\tau_\gamma X_p, \tau_\gamma Y_p) \circ \tau_\gamma$$

where X_p, Y_p *run through* $T_p\mathbb{M}$ *and* γ *runs through all paths starting at* p*.*

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Theorem (Berger)

Suppose $(\mathbb{M}, g)$ *is a Riemannian manifold such that* $\mathrm{Hol}^0(\mathbb{M})$ *is irreducible. Then at least one of the following is satisfied:*

- $\mathrm{Hol}^0(\mathbb{M})$ *acts transitively on the sphere, or*
- $\mathbb{M}$ *is a locally symmetric space of rank greater than or equal to 2.*

Symmetric Case

Definition

A Riemannian manifold $\mathbb{M}$ is called symmetric if for each point $p \in \mathbb{M}$ there exists an isometry $f_p \colon \mathbb{M} \to \mathbb{M}$ such that

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- $T_p(f_p) = -Id_{T_p\mathbb{M}}$

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There is a complete classification due to Cartan of all symmetric Riemannian spaces.

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- *The manifold is a homogenous space $\mathbb{M} = G/H$ where G is a Lie group, and H is a compact subgroup of G.*

This relates to the holonomy group in the following way:

Theorem

For an irreducible simply connected symmetric manifold $\mathbb{M} = G/H$, we have that $\mathrm{Hol}(\mathbb{M}) = H$ acting by the adjoint representation.

Nonsymmetric Case

Theorem (Berger, Simons)

For an irreducible simply connected nonsymmetric manifold $\mathbb{M}$*, one of the following cases occurs:*

$dim(\mathbb{M})$	$\mathrm{Hol}^0(\mathbb{M})$	*Type*
n	$O(n)$	*Generic*
n	$SO(n)$	*Oriented*
$n = 2m$	$U(m)$	*Kähler*
$n = 2m$	$SU(m)$	*Calabi-Yau*
$n = 4m$	$Sp(m) \cdot Sp(1)$	*Quaternion-Kähler*
$n = 4m$	$Sp(m)$	*Hyperkähler*
$n = 7$	G_2	
$n = 8$	$Spin(7)$	

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4. Since $Z = 0$, $(\nabla R)_p = 0$. This can be easily extended to $\nabla R = 0$ everywhere, and so $\mathbb{M}$ is symmetric.

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- The holonomy of a manifold is closely related to its curvature tensor R, and so it is possible to consider the possible curvatures given a holonomy group, or vice-versa.
- It's an open question whether $\mathrm{Hol}(\mathbb{M})$ can be noncompact for compact manifolds $\mathbb{M}$.