Comparison Theorems on H-type Foliations

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Basic Definitions

Let $\mathbb{M}$ be a smooth manifold. We say that

- $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$ is a sub-Riemannian manifold if
  - $\mathcal{H}$ is a constant rank, bracket generating subbundle of $T\mathbb{M}$,
  - and $g_{\mathcal{H}}$ is a inner product on $\mathcal{H}$.
- $(\mathbb{M}, \mathcal{H}, g)$ is a sub-Riemannian manifold with metric preserving complement or sRmc-manifold if
  - $(\mathbb{M}, g)$ is a Riemannian manifold,
  - the metric orthogonally splits as $g = g_{\mathcal{H}} \oplus g_{\mathcal{V}}$,
  - and $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$ is a sub-Riemannian manifold.

  We denote by $\mathcal{V}$ the orthogonal complement of $\mathcal{H}$ by g.

Riemannian Comparisons

In Riemannian geometry it is a well known special case of the Rauch comparison theorem that for a Riemannian manifold $(\mathbb{M}, g)$ with Ricci curvature not less than $(n-1)\kappa$ that the distance function $r(\cdot) = d(x, \cdot)$ satisfies

$$\Delta r \leq \begin{cases} (n-1)\sqrt{\kappa}\cot(\sqrt{\kappa}r) & \kappa > 0 \\ \frac{n-1}{r} & \kappa = 0 \\ (n-1)\sqrt{|\kappa|}\coth(\sqrt{|\kappa|}r) & \kappa < 0 \end{cases}$$

This is intimately related with diameter estimates and Hessian comparisons. In this work, we recover similar estimates on a class of sub-Riemannian manifolds.

Motivating Example: Hopf Fibration

Consider $\mathbb{S}^{2n+1}$ foliated as

$$\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1} \xrightarrow{\pi} \mathbb{C}P^n$$

Defining the horizontal distribution as

$$\mathcal{H} = d\pi^{-1}(T\mathbb{C}P^n)$$

the space $(\mathbb{S}^{2n+1}, \mathcal{H}, g)$ is an sRmc-manifold, foliated with leaves

$$\mathcal{V}_x \cong \mathbb{S}^1$$

Gromov-Hausdorff Convergence

For a sRmc-manfiold $(\mathbb{M}, \mathcal{H}, g)$ we define the canonical variation of the metric

$$g_\varepsilon = g_\mathcal{H} + \frac{1}{\varepsilon} g_\mathcal{V}$$

which in the Gromov-Hausdorff sense

$$(\mathbb{M}, \mathcal{H}, g_\varepsilon) \xrightarrow{\varepsilon \to 0^+} (\mathbb{M}, \mathcal{H}, g_\mathcal{H})$$

Any sub-Riemannian manifold can be constructed this way; our goal is to leverage our knowledge of the Riemannian structure to understand the sub-Riemannian one.

Hladky-Bott Connection

Theorem (Hladky '12 [5])

There exists a unique metric connection ∇ *on* $(\mathbb{M}, \mathcal{H}, g)$ *such that*

1. $\mathcal{H}$ *and* $\mathcal{V}$ *are* ∇*-parallel,*
2. *The torsion* T *of* ∇ *satisfies*
   - $T(\mathcal{H}, \mathcal{H}) \subset \mathcal{V}$,
   - $T(\mathcal{V}, \mathcal{V}) \subset \mathcal{H}$
3. *For every* $X, Y \in \Gamma(\mathcal{H}), Z, V \in \Gamma(\mathcal{V})$,
   - $\langle T(X, Z), Y \rangle_{\mathcal{H}} = \langle T(Y, Z), X \rangle_{\mathcal{H}}$
   - $\langle T(Z, X), V \rangle_{\mathcal{V}} = \langle T(V, X), Z \rangle_{\mathcal{V}}$.

This is called the Hladky-Bott connection.

Hladky-Bott Connection

We can explicitly write ∇ in terms of the Levi-Civita connection ∇^g as

$$\nabla_X Y = \begin{cases} \pi_{\mathcal{H}} \nabla^g_X Y & X, Y \in \Gamma(\mathcal{H}) \\ \pi_{\mathcal{H}}[X, Y] + A_X Y & Y \in \Gamma(\mathcal{H}), X \in \Gamma(\mathcal{V}) \\ \pi_{\mathcal{V}}[X, Y] + A_X Y & Y \in \Gamma(\mathcal{V}), X \in \Gamma(\mathcal{H}) \\ \pi_{\mathcal{V}} \nabla^g_X Y & X, Y \in \Gamma(\mathcal{V}) \end{cases}$$

where the tensor A is defined by

$$\langle A_X Y, Z \rangle = \frac{1}{2}\left((\mathcal{L}_{X_{\mathcal{V}}} g)(Y_{\mathcal{H}}, Z_{\mathcal{H}}) + (\mathcal{L}_{X_{\mathcal{H}}} g)(Y_{\mathcal{V}}, Z_{\mathcal{V}}) \right)$$

J Map

On $(\mathbb{M}, \mathcal{H}, g)$ we can associate to each vector field $Z \in \Gamma(T\mathbb{M})$ an endomorphism J_Z of $T\mathbb{M}$ defined by

$$\langle J_Z X, Y \rangle = \langle Z, T(X, Y) \rangle$$

If $\mathcal{V}$ is integrable,

$$\begin{cases} J_Z X \in \mathcal{H} & \text{if } Z \in \mathcal{V}, X \in \mathcal{H} \\ J_Z X = 0 & \text{otherwise} \end{cases}$$

We thus take the perspective

$$J \colon \mathcal{V} \to \mathrm{End}(\mathcal{H}), \quad Z \mapsto J_Z$$

H-type Foliations

Definition

Let $(\mathbb{M}, \mathcal{H}, g)$ be a sRmc-manfiold. We say that $(\mathbb{M}, \mathcal{H}, g, J)$ is an <u>H-type foliation</u> if

1. $\mathcal{V}$ is integrable, and
2. for all $X, Y \in \Gamma(\mathcal{H}), Z \in \Gamma(\mathcal{V})$,

$$\langle J_Z X, J_Z Y\rangle_{\mathcal{H}} = \|Z\|^2 \langle X, Y\rangle_{\mathcal{H}}$$

Parallel Torsion

We also refine the definition of H-type foliations based on the behavior of derivatives of the Hladky-Bott torsion T.

1. If $\delta_{\mathcal{H}} T = 0$ we say $\mathbb{M}$ is of Yang-Mills type,
2. If $\nabla_{\mathcal{H}} T = 0$ we say $\mathbb{M}$ has horizontally parallel torsion, and
3. If $\nabla T = 0$ we say $\mathbb{M}$ has completely parallel torsion.

Lemma

All H-type foliations are Yang-Mills.

Example: Twistor Spaces

Let $(\mathbb{M}, g)$ be a $4n$-dimensional $(n \geq 2)$ quaternionic-Kähler manifold, and fix a quaternionic structure E spanned by $\mathcal{I}, \mathcal{J}, \mathcal{K} \in \mathrm{End}(T\mathbb{M})$. Choosing a metric on E so that $\mathcal{I}, \mathcal{J}, \mathcal{K}$ are orthonormal, we define the twistor space over $\mathbb{M}$ to be the unit sphere bundle of E. In this case, we have

- $\mathcal{H}_x \cong T_x\mathbb{M}$,
- $\mathcal{V}_x \cong \mathbb{C}P^1$.

Here $m = \dim \mathcal{V} = 2$ and there is a quaternionic structure induced by $\mathcal{V}$ acting on $\mathcal{H}$.

Further Examples of H-type foliations

rk$(\mathcal{V})$	**rk**$(\mathcal{H})$	$\mathbb{M}$	Torsion
any	2k	Heisenberg Group	C. Parallel
1	2k	Mixed K-contact	Yang-Mills
		Sasakian/Hopf fibration	C. Parallel
2	4k	Salamon Twistor Spaces	H. Parallel
3	4k	Mixed 3K-contact	Yang-Mills
		3-Sasakian/Quaternionic Hopf	H. Parallel
		Torus Bundles over HK	C. Parallel
4,5,7	8k	Grassmannian Type	H. Parallel
7	8	Octonionic Hopf	H. Parallel

Dimensional Restrictions

For unit $Z \in \mathcal{V}$, the maps J_Z will induce complex, quaternionic, and octonionic structures on $\mathcal{H}$. As a consequence,

Lemma

Denote $m = \mathbf{rk}(\mathcal{V})$, $n = \mathbf{rk}(\mathcal{H})$. Then

1. $m \leq n - 1$,
2. $m = n - 1$ *implies* $n = 2, 4,$ *or* 8,
3. $n = 2k$, *and furthermore*
   - *if* $m \geq 2$ *then* $n = 4k$,
   - *if* $m \geq 4$ *then* $n = 8k$.

Clifford Structures

Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation with $Z_i, Z_j \in \mathcal{V}$, then

$$J_{Z_i} J_{Z_j} + J_{Z_j} J_{Z_i} = -2\langle Z_i, Z_j \rangle Id_{\mathcal{H}}$$

and so we can extend J in the natural way to

$$J \colon Cl(\mathcal{V}) \to \mathrm{End}(\mathcal{H})$$

There is a classification of such Clifford structures over Riemannian manifolds. (A. Moroianu, U. Semmelmann '11 [6])

Parallel Horizontal Clifford Structures

Arising from Riemannian or semi-Riemannian foliations with curvature constancy we have the following notion:

Definition

Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation with horizontally parallel torsion. Then if there exists a map

$$\Psi \colon \mathcal{V} \times \mathcal{V} \to Cl_2(\mathcal{V})$$

such that

$$(\nabla_{Z_1} J)_{Z_2} = J_{\Psi(Z_1, Z_2)}$$

for all $Z_1, Z_2 \in \Gamma(\mathcal{V})$ then we say that $\mathbb{M}$ has a parallel horizontal Clifford structure.

This has a strong relationship with the horizontal holonomy of the space.

Parallel Horizontal Clifford Structures

Moreover, these structures are rigid in that they must be of the following form:

Lemma (Baudoin, Grong, Rizzi, & M. '18 [2])

Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation with parallel horizontal Clifford structure. There exists $\kappa \in \mathbb{R}$ such that

$$\Psi(Z_1, Z_2) = \kappa(Z_1 \cdot Z_2 + \langle Z_1, Z_2 \rangle)$$

for all $Z_1, Z_2 \in \Gamma(\mathcal{V})$; moreover the sectional curvature of the leaves associated to $\mathcal{V}$ is constant and equal to κ^2.

J^2 condition

In many model cases, such as the Complex-, Quaternionic-, and Octonionic-Hopf fibrations we have a stronger property:

Definition

Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation. We say that it satisfies the J^2 condition if for every $Z_1, Z_2 \in \mathcal{V}$ with $\langle Z_1, Z_2 \rangle = 0$ there exists $Z_3 \in \mathcal{V}$ such that

$$J_{Z_1} J_{Z_2} = J_{Z_3}$$

The H-type groups with this property were classified by (M. Cowling, A.H. Dooley, A. Korányi, and F.Ricci '91 [4])

Metric Connections, Jacobi Equation

For the remainder of the talk, let $(M, \mathcal{H}, g_\varepsilon)$

- be a sRmc-manifold,
- equipped with the canonical variation g_ε,
- having horizontally parallel torsion,
- and satisfying the J^2 condition.

For any $\varepsilon > 0$ the connection

$$\hat{\nabla}^\varepsilon_X Y = \nabla_X Y + J^\varepsilon_X Y$$

will be metric with metric adjoint ∇^ε. For a g_ε-geodesic γ, the Jacobi equation in this setting is

$$\hat{\nabla}^\varepsilon_{\dot\gamma} \nabla^\varepsilon_{\dot\gamma} W + \hat{R}^\varepsilon(W, \dot\gamma)\dot\gamma = 0$$

The Comparison Principle

Theorem (Baudoin, Grong, Kuwada, & Thalmaier '17 [1])

- *Let* $x, y \in \mathbb{M}$,
- $\gamma \colon [0, r_\varepsilon] \to \mathbb{M}$ *a unit speed* g_ε*-geodesic connecting* x, y, *and*
- $W_1, \cdots, W_k$ *be a collection of vector fields along* γ *such that*

$$\sum_{i=0}^{k} \int_0^{r_\varepsilon} \langle \hat{\nabla}^\varepsilon_{\dot\gamma} \nabla^\varepsilon_{\dot\gamma} W_i + \hat{R}^\varepsilon(W_i, \dot\gamma)\dot\gamma, W_i \rangle_\varepsilon \geq 0$$

then at $y = \gamma(r_\varepsilon)$ *it holds that*

$$\sum_{i=0}^{k} \mathrm{Hess}^{\hat{\nabla}^\varepsilon}(r_\varepsilon)(W_i, W_i) \leq \sum_{i=0}^{k} \langle W_i, \hat{\nabla}^\varepsilon_{\dot\gamma} W_i \rangle_\varepsilon$$

with equality if and only if the W_i *are Jacobi fields.*

Horizontal Splitting

We introduce an orthogonal splitting of the horizontal bundle. Fixing a vector field $Y \in \mathcal{H}$,

$$\mathcal{H} = \mathrm{span}(Y) \oplus \mathcal{H}_{Riem}(Y) \oplus \mathcal{H}_{Sas}(Y)$$

where

$$\mathcal{H}_{Sas}(Y) = \{J_Z Y | Z \in \mathcal{V}\}$$

$$\mathcal{H}_{Riem}(Y) = \{X \in \mathcal{H} | X \perp \mathcal{H}_{Sas} \oplus \mathrm{span}(Y)\}$$

Lemma

Denoting $n = \mathbf{rk}(\mathcal{H}), m = \mathbf{rk}(\mathcal{V})$, *we will have*

$$\dim(\mathcal{H}_{Sas}) = m, \quad \dim(\mathcal{H}_{Riem}) = n - m - 1$$

Comparison Functions

Simliarly to the Riemannian case, we consider the comparision functions

$$F_{Riem}(r,\kappa) = \begin{cases} \sqrt{\kappa}\cot(\sqrt{\kappa}r) & \text{if } \kappa > 0 \\ \frac{1}{r} & \text{if } \kappa = 0 \\ \sqrt{|\kappa|}\coth(\sqrt{|\kappa|}r) & \text{if } \kappa < 0 \end{cases}$$

$$F_{Sas}(r,\kappa) = \begin{cases} \frac{\sqrt{\kappa}\left(\sin(\sqrt{\kappa}r)-\sqrt{\kappa}r\cos(\sqrt{\kappa}r)\right)}{2-2\cos(\sqrt{\kappa}r)-\sqrt{\kappa}r\sin(\sqrt{\kappa}r)} & \text{if } \kappa > 0 \\ \frac{4}{r} & \text{if } \kappa = 0 \\ \frac{\sqrt{\kappa}\left(\sqrt{\kappa}r\cosh(\sqrt{\kappa}r)-\sinh(\sqrt{\kappa}r)\right)}{2-2\cosh(\sqrt{\kappa}r)+\sqrt{\kappa}r\sinh(\sqrt{\kappa}r)} & \text{if } \kappa < 0 \end{cases}$$

These comparison functions will correspond to the splitting of $\mathcal{H}$.

Hessian Comparisons

Theorem (Baudoin, Grong, Rizzi, & M. '19 [3])

1. *Let* $\gamma : [0, r_\varepsilon] \to \mathbb{M}$ *be a* g_ε*-geodesic. Then*

$$\mathrm{Hess}(r_\varepsilon)(\dot\gamma, \dot\gamma) \le \frac{\|\dot\gamma\|^2 \left(1 - \|\dot\gamma\|^2\right)}{r_\varepsilon}$$

2. *If* $\mathrm{Sec}(X \wedge Y) \ge \rho > 0$ *for all unit* $X, Y \in \mathcal{H}_{Riem}(\dot\gamma)$, *then*

$$\mathrm{Hess}(r_\varepsilon)(X, X) \le F_{Riem}(r_\varepsilon, K)$$

3. *If* $\mathrm{Sec}(X \wedge J_Z X) \ge \rho > 0$ *for all unit* $X \in \mathcal{H}_{Sas}(\dot\gamma)$, *then*

$$\mathrm{Hess}(r_\varepsilon)(X, X) \le F_{Sas}(r_\varepsilon, K)$$

Where K *is a constant depending on* $\rho, \varepsilon, \|\nabla_{\mathcal{V}} r_\varepsilon\|$, *and* $\|\nabla_{\mathcal{H}} r_\varepsilon\|$.

Horizontal Ricci Curvature

We can define the horizontal Ricci curvature as the trace of the Riemann tensor,

$$\begin{aligned}\mathrm{Ric}_{\mathcal{H}}(X,X) &= \sum_{i=0}^{n} \langle R^{\nabla}(W_i,X)X, W_i\rangle \\ &= \langle R^{\nabla}(Y,X)X, Y\rangle + \mathrm{Ric}_{Sas}(X,X) + \mathrm{Ric}_{Riem}(X,X)\end{aligned}$$

where the splitting corresponds to the decomposition

$$\mathcal{H} = \mathrm{span}(Y) \oplus \mathcal{H}_{Sas} \oplus \mathcal{H}_{Riem}$$

Diameter Estimates

Theorem (Baudoin, Grong, Rizzi, & M. '19 [3])

Let $\rho > 0$. Then for unit $X \in \mathcal{H}$,

1. $\dfrac{\mathrm{Ric}_{Riem}(X,X)}{n-m-1} \geq \rho \implies \mathrm{diam}_0(\mathbb{M}) \leq \dfrac{\pi}{\sqrt{\rho}}$
2. $\mathrm{Sec}(X \wedge J_Z X) \geq \rho \implies \mathrm{diam}_0(\mathbb{M}) \leq \dfrac{2\pi}{\sqrt{\rho}}$
3. $\dfrac{\mathrm{Ric}_{Sas}(X,X)}{m} \geq \rho \implies \mathrm{diam}_0(\mathbb{M}) \leq \dfrac{2\pi\sqrt{3}}{\sqrt{\rho}}$

and in each case the fundamental group of $\mathbb{M}$ must be finite.

The first two of these are sharp, as they are achieved in the complex, quaternionic, and octonionic Hopf fibrations.

sub-Laplacian

Similarly to the horizontal Ricci curvature, we can define the sub-Laplacian as the trace of the Hessian. For the distance function r_ε along a geodesic γ with $Y = \nabla_{\mathcal{H}} r_\varepsilon$,

$$\Delta_{\mathcal{H}} r_\varepsilon = \sum_{i=0}^{n} \mathrm{Hess}(r_\varepsilon)(W_i, W_i)$$

$$= \mathrm{Hess}(r_\varepsilon)(Y, Y) + \sum_{i=0}^{m} \mathrm{Hess}(r_\varepsilon)(J_{Z_i} Y, J_{Z_i} Y) + \sum_{i=0}^{n-m-1} \mathrm{Hess}(r_\varepsilon)(W_i, W_i)$$

for appropriate bases $\{W_i\}$ of $\mathcal{H}$ and $\{Z_i\}$ of $\mathcal{V}$. This splitting corresponds again to the decomposition

$$\mathcal{H} = \mathrm{span}(Y) \oplus \mathcal{H}_{Sas} \oplus \mathcal{H}_{Riem}$$

Laplacian Comparisons

In each component of the horizontal decomposition we can use the previous comparisons on the Hessian to obtain

Theorem (Baudoin, Grong, Rizzi, & M. '19 [3])

Let $(\mathbb{M}, g, \mathcal{H})$ be an H-type foliation with parallel horizontal Clifford structure and satisfying the J^2 condition, and with nonnegative horizontal Bott curvature. Then there exists a $C > 4$ such that

$$\Delta_{\mathcal{H}} r_0 \leq \frac{n - m + 3 + C(m-1)}{r_0}$$

This is not sharp, but we can recover sharp estimates in each subspace.

Future work

- Relationship between horizontal holonomy groups and horizontal parallel Clifford structures
- Index theoretic results extending earlier work on contact manifolds
- Study of connections respecting foliations and sub-Riemannian structures

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Thank you for your attention!