Connections on Manifolds: An Invitation to Riemannian Geometry

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First Ideas

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- First axiomatic study: Euclid's Elements
- Gave a foundation from which geometry was studied for thousands of years
- With coordinates (deCarte) we have $\mathbb{R}^n$

The Problem of the Parallel Postulate

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- The intent of the postulate is clear,
- But was always considered to be too complicated to be a natural axiom.
- However, no proof of it could be found starting only from the other axioms.
- Mathematicians (starting with Euclid) struggled for thouands of years to resolve this.

Non-Euclidean Geometry

The resolution comes in the form of a counterexample:

Example

Consider the geometry (points, lines, etc.) lying on a sphere. These satisfy all of the axioms of Euclidean geometry except for the Parallel Postulate.

This put to rest the question of naturalness of the Parallel Postulate, and began the study of non-Euclidean geometry.

Curvature

The study of curvature began by consideration of lines:

Definition

Given an arclength parametrization of a curve $\gamma\colon [0,1] \to \mathbb{R}^n$, its curvature is

$$\kappa_\gamma(s) = \|\vec{T}'(s)\|$$

where $\vec{T}(s) = \frac{\gamma'(s)}{\|\gamma'(s)\|}$ is its unit tangent vector.

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- This quantity measures the deviation of a plane curve from a straight line.
- This fully quantifies curvature for curves, but it does not extend well to surfaces.

Curvature of surfaces

Let $\mathbb{M}$ be a surface in $\mathbb{R}^3$, and fix a point $x \in \mathbb{M}$.

- We say γ is a curve in $\mathbb{M}$ if $\gamma(t) \in \mathbb{M}$ for all $t \in [0,1]$, and denote $\Gamma_x = \{$all curves γ in $\mathbb{M}$ such that $\gamma(0) = x\}$.

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- Different curves in Γ_x can have different curvature at x.
- If we fix a curve $\gamma \in \Gamma_x$ and change the coordinate system (even isometrically), the curvature of γ at x may change.
- Define the principal curvatures of $\mathbb{M}$ at x:

$$k_1(x) = \sup\{\kappa_\gamma(0) \colon \gamma \in \Gamma_x\}$$
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Under an isometric change of coordinate system the principal curvatures at x can change.

Gauss' Theorema Egregium

The fundamental breakthrough was realized by Gauss.

Theorem (Theorema Egregium (1828))

Consider a surface $\mathbb{M}$ isometerically embedded in $\mathbb{R}^3$, and fix a point $x \in \mathbb{M}$. Then the quantity

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- This shows that there are properties of surfaces that are intrinsic, that is they don't depend on the ambient space.
- It became a core goal of geometry to understand intrinsic properties of geometric objects.

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- Smooth Manifolds: Derivatives of functions are well-defined
- Riemannian Manifolds: Distance, length, and angle are well-defined

Topological Spaces

Given a set $\mathbb{M}$, we define a topology $\mathcal{T}$ on $\mathbb{M}$ to be a subset of the power set of $\mathbb{M}$ satisfying

1. $\mathbb{M} \in \mathcal{T}$
2. For any collection of elements of $\mathcal{T}$, their union is in $\mathcal{T}$
3. For any finite collection of elements of $\mathcal{T}$, their intersection is in $\mathcal{T}$.

We call the pair $(\mathbb{M}, \mathcal{T})$ a topological space and the elements of $\mathcal{T}$ open sets.

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- Locally Euclidean: Every point has a neighborhood that is homeomorphic to $\mathbb{R}^n$.

A topological space $(\mathbb{M}, \mathcal{T})$ that has all of these properties is called a topological manifold.

Study of topological manifolds

Already at this level there is a wealth of intrinsic information (invariant under homeomorphism). This includes properties such as

- Compactness
- Connectedness
- Fundamental Group and higher Homotopy Groups

This kind information is fundamental to any further structure built on the space, and so detecting it is a core goal of differential geometry.

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$$\phi_U|_E \circ \phi_V^{-1}|_{\phi_V(E)} \colon \phi_V(E) \to \phi_U(E)$$

a <u>transition map</u>.

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Definition

If there is a collection $\mathcal{F}$ of smooth (infinitely differentiable) transition maps that cover $\mathbb{M}$, we call $(\mathbb{M}, \mathcal{T}, \mathcal{F})$ a smooth manifold.

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- Fix a point $x \in \mathbb{M}^n$; at this point "attach" a copy of $\mathbb{R}^n$ at the origin (think tangent planes to surfaces). We call these tangent spaces $T_x\mathbb{M}$, and their disjoint union the tangent bundle $T\mathbb{M} = \cup_{x\in\mathbb{M}} T_x\mathbb{M}$.

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- The vectors in $T_x\mathbb{M}$ can be put into 1-1 correspondence with the smooth curves $\gamma \colon [0,1] \to \mathbb{M}$ passing through x "in the same direction", so we view vectors in $T_x\mathbb{M}$ as tangent vectors to curves.

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- If at every point x in $\mathbb{M}$ we choose a tangent vector X_x, we call it a vector field X.

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- So we can identify a vector in $T_x\mathbb{M}$ called the gradient

$$\nabla f(x) = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}$$

where $\frac{\partial}{\partial x_i}$ is the vector associated to the x_i-direction.

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- To recover this, we will introduce an inner product (that is, a symmetric bilinear map)

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on every tangent space $T_x\mathbb{M}$.

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  on every tangent space $T_x\mathbb{M}$.
- If these inner products vary smoothly as a function of x, we call it a <u>Riemannian metric</u> denoted g.
- A smooth manifold with a Riemannian metric $(\mathbb{M}, \mathcal{T}, \mathcal{F}, g)$ is called a <u>Riemannian manifold</u>.

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- The length of a curve $\gamma\colon [0,1] \to \mathbb{M}$ is given by the usual arclength formula computed using its velocity vectors γ'

$$L(\gamma) = \int_0^1 \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}\, dt$$

Parallel Transport Problem

Suppose we have two points $x, y \in \mathbb{M}$, and a vector $X_x \in T_x\mathbb{M}$. By construction,

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But what vector in $T_y\mathbb{M}$ should correspond to X_x?

We can visualize connecting x and y by a curve, and transporting X_x along the curve to y, but it's still not clear how this should work.

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Any such ∇ is called a connection.

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Given a curve γ from x to y and a γ-parallel vector field X, we say that X_y is the parallel transport of X_x along γ.

Levi-Civita Connection

While there are many possible connections on a manifold, there is one in particular that we are interested in.

Theorem

There exists a unique connection ∇ *such that*

1. $\nabla_X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$
2. $\nabla_X Y - \nabla_Y X = [X, Y]$

This connection is called the Levi-Civita connection.

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4. Scalar Curvature:

$$K = Tr_g(Ric)$$

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- Nash Embedding Theorem: if $\mathbb{M}$ is compact,

$$\mathbb{M} \hookrightarrow \mathbb{R}^n, \quad n \leq \frac{m(3m+11)}{2}$$

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$$\Delta r \leq \begin{cases} (n-1)\sqrt{\kappa}\cot(\sqrt{\kappa} r) & \kappa > 0 \\ \frac{n-1}{r} & \kappa = 0 \\ (n-1)\sqrt{|\kappa|}\coth(\sqrt{|\kappa|} r) & \kappa < 0 \end{cases}$$

- Bonnet-Meyers Diameter Estimates: If $\kappa > 0$ then

$$\text{diam}(\mathbb{M}) \leq \frac{\pi}{\sqrt{\kappa}}$$

and the fundamental group of $\mathbb{M}$ must be finite.

Other Geometries

- Symplectic
- Kähler
- Contact
- Pseudo-Riemannian