Confoliations: An Interpolation of Classical Topological Structures

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Motivation

- There is great interest in the geometry of surfaces, submanifolds, and contact structures.
- In particular, the study of stochastic processes (especially Brownian motion and heat flows) are active areas of research.
- Today I will discuss these various structures, and introduce a (lesser known) structure designed to unify them.

The Setting

In this talk, $\mathbb{M}$ will always be a connected smooth manifold. To review, we have the following structures:

- Topological manifold
- Smooth structure

Topological Manifold

A topological space $(\mathbb{M}, \mathcal{T})$ is called a topological manifold if it has all of the following properties:

1. Hausdorff
2. Second-Countable
3. Locally Euclidean

Smooth manifolds

- Let $(\mathbb{M}, \mathcal{T})$ be a topological manifold.
- Let $U, V \in \mathcal{T}$ be homeomorphic to $\mathbb{R}^n$ by maps ϕ_U, ϕ_V, and suppose $U \cap V \neq \emptyset$; denote $E = U \cap V$.
- We call the continuous map

$$\phi_U|_E \circ \phi_V^{-1}|_{\phi_V(E)} : \phi_V(E) \to \phi_U(E)$$

a transition map.

Definition

If there is a collection Φ of smooth (infinitely differentiable) transition maps that cover $\mathbb{M}$, we call $(\mathbb{M}, \mathcal{T}, \Phi)$ a smooth manifold.

The Tangent Bundle

- Associated to every point $p \in \mathbb{M}$ we have a copy of $\mathbb{R}^n$. We call these tangent spaces $T_p\mathbb{M}$, and their disjoint union the tangent bundle $T\mathbb{M} = \cup_{p\in\mathbb{M}} T_p\mathbb{M}$.
- The vectors in $T_p\mathbb{M}$ can be put into 1-1 correspondence with the smooth curves $\gamma\colon [0,1] \to \mathbb{M}$ passing through p "in the same direction", so we view elements of $T_p\mathbb{M}$ as tangent vectors to curves.
- If at every point p in $\mathbb{M}$ we choose a tangent vector $X_p \in T_p\mathbb{M}$, we call the collection a vector field $X \in \mathfrak{X}(\mathbb{M})$.

The Exterior Derivative

- We need a generalization of the notion of directional derivative.
- Given a smooth function f, we can define its exterior derivative $df \in T^*\mathbb{M}$ in local coordinates by

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i$$

where dx^i is 1 on vector fields in the x^i direction.

The Exterior Derivative

- We call the space $\Lambda^k T^*\mathbb{M}$ the space of k-forms. That is, k-forms are alternating products of the cotangent bundle.
- We can now think of f as a 0-form df as a 1-form, and

$$d\colon \Lambda^0 T^*\mathbb{M} \to \Lambda^1 T^*\mathbb{M}$$

- Now let $\alpha \in \Lambda^s T^*\mathbb{M}, \beta \in \Lambda^t T^*\mathbb{M}$. We can define the exterior derivative

$$d\colon \Lambda^k T^*\mathbb{M} \to \Lambda^{k+1} T^*\mathbb{M}$$

by

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$$

Submanifolds

- Given two smooth manifolds $\mathbb{M}, \mathbb{N}$, $\dim \mathbb{M} = m, \dim \mathbb{N} = n$, we say that $\mathbb{N}$ is a submanifold of $\mathbb{M}$ if there is a map $f\colon N \to M$ that is an immersion. That is the differential

$$d_p f\colon T_p\mathbb{N} \to T_{f(p)}\mathbb{M}$$

is injective.
- Certainly, $n \leq m$ for $\mathbb{N}$ to be a submanifold of $\mathbb{M}$.
- We can visualize submanifolds $\mathbb{N}$ as actually existing inside the larger manifold $\mathbb{M}$.

Foliations

- If it is possible to decompose a manifold $\mathbb{M}$ into a collection of submanifolds all of the same dimension n, we call the collection of submanifolds a foliation of $\mathbb{M}$.
- A single element of the collection of submanifolds is called a leaf of the foliation.
- We call the number $m - n$ the codimension of the foliation.

Submersion Construction

- Given two manifolds $\mathbb{M}, \mathbb{N}$, $m \geq n$, and a map $f\colon \mathbb{M} \to \mathbb{N}$, we call f a <u>submersion</u> if the rank of the differential is full, that is $\mathrm{rk}(f) = n$.
- In this case, f defines a codimension n foliation of $\mathbb{M}$ by submanifolds $f^{-1}(p)$ for each $p \in \mathbb{N}$.
- Consider the submersion $f\colon \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f\colon (x, y, z) \mapsto z$$

The leaves of the induced foliation are the planes parallel to the xy-plane.

Example: Reeb Foliation

An interesting example of a nontrivial foliation:

- First, we define a submersion

$$\begin{cases} f\colon D^2 \times \mathbb{R} \to \mathbb{R} \\ (r, \theta, t) \mapsto (r^2 - 1)e^t \end{cases}$$

- The induced foliation is invariant by the map

$$(r, \theta, t) \mapsto (r, \theta, t + a)$$

  for integers a, and so the foliation can be thought of as on the solid torus $(D^2 \times \mathbb{R})/\mathbb{Z}$.
- We can see S^3 as being the union of two disjoint solid tori with the usual torus as their mutual boundary, so we have a foliation of S^3.

Holonomy of Foliations

- Given a foliation and a loop γ contained entirely in one leaf, we can try to understand the global structure of the foliation by considering the holonomy of γ.
- Extend the loop γ to an annulus $A = \gamma \times [0, 1]$, requiring that A be transversal to the foliation. Then A itself will be foliated, with leaves diffeomorphic to γ.
- The holonomy of γ is the germ at 0 of the Poincaré return map, which is to say a measure of how much the leaves "spread apart".
- We say a leaf of a foliation has no holonomy if the holonomy is trivial along any loop, and we say the leaf has trivial linear holonomy if the differential of the holonomy map at 0 is different from 1.

Tangent Distribution

- Any foliation can be characterized by its associated tangent distribution.
- That is, at every point p of the manifold, the tangent space p has a linear subspace determined by the tangent space of the leaf through p.
- Such a collection of linear subspaces of the tangent spaces on a manifold is called a distribution and will be denoted by ξ.
- A distribution ξ can be expressed by saying there exists a 1-form α such that the Pfaffian condition $\alpha(\xi) = 0$ is satisfied.

Integrable Distributions

- A distribution that is everywhere tangent to a submanifold is called integrable.
- Clearly, the distribution associated to a foliation is integrable, as it is tangent to the leaves (which are submanifolds by definition.)
- Given an integrable distribution ξ, we can recover a foliation. This is equivalent to solving a system of PDEs.

Contact Manifolds

- A 2n+1-dimensional manifold $\mathbb{M}$ with a 1-form α such that

$$\alpha \wedge (d\alpha)^n > 0$$

is called a contact manifold, and α is called a contact form.
- There exists a unique vector field η called the Reeb vector field on contact manifolds such that

$$\begin{cases} \alpha(\eta) = 1 \\ d\alpha(\eta) = 0 \end{cases}$$

Symplectic Manifolds

- A 2n-dimensional manifold $\mathbb{M}$ with a nondegenerate 2-form ω such that is called a symplectic manifold.
- Symplectic manifolds are generalizations of complex geometry in the sense that there always exist complex structures $J \in \mathrm{End}(T\mathbb{M})$ such that

$$\begin{cases} J^2 = -Id \\ \omega(JX, JY) = \omega(X, Y) \end{cases}$$

Contactization and Symplectization

- From a symplectic manifold $\mathbb{M}$ with exact symplectic form $\omega = d\lambda$ we can construct a 2n+1-dimensional contact manifold

$$P = \mathbb{M} \times \mathbb{R}$$

with contact form

$$\alpha = \pi^* \lambda + dt$$

- In the other direction, from a 2n+1-dimensional contact manifold $\mathbb{M}$ with contact form α we can construct a 2n+2 dimensional symplectic manifold

$$P = \mathbb{M} \times \mathbb{R}^+$$

with symplectic form

$$\omega = d(e^t \alpha) = e^t (d\alpha - \alpha \wedge dt)$$

Example: Heisenberg Group

- The Heisenberg Group can be defined as $\mathbb{R}^3$ equipped with the 1-form
$$\alpha = \frac{1}{2}(xdy - ydx) + dz$$
- At every point of $\mathbb{R}^3$ there is a plane defined by the Pfaffian $\alpha = 0$. These planes are spanned by the vector fields
$$X = \frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial z}$$
$$Y = \frac{\partial}{\partial y} - \frac{1}{2}x\frac{\partial}{\partial z}$$

Contact Distributions

- Since contact manifolds are defined by 1-forms α, it is natural to consider the distribution ξ defined by

$$\alpha(\xi) = 0$$

- The distribution ξ is nowhere integrable, which is to say that there is no point in the manifold where it is (even locally) tangent to a submanifold.
- Given a Riemannian metric g, there is a natural structure J that is almost-complex (that is $J^2 = -Id$) and is nonvanishing strictly on ξ. this gives some intuition to the proceedures of contactization and symplectification.

Frobenius Theorem

- The integrability of distributions is apparently key to understanding both foliations and contact structures.
- We have an explicit characterization of the integrability of distributions due to Frobenius. We state a version in dimension 3, written in terms of 1-forms.

Theorem (Frobenius Theorem)

Suppose that a distribution ξ on a dimension 3 manifold is defined by a 1-form α and the Pfaffian condition

$$\alpha(\xi) = 0.$$

Then ξ is integrable if and only if

$$\alpha \wedge d\alpha \equiv 0.$$

Confoliations

- So, on a dimension 3 manifold we can see contact structures

$$\alpha \wedge d\alpha > 0$$

  and foliations

$$\alpha \wedge d\alpha = 0$$

  as somehow opposite.

- This motivates an interpolation: a distribution ξ associated to a 1-form α by the usual Pfaffian equation is called a <u>confoliation</u> if

$$\alpha \wedge d\alpha \geq 0$$

Characterization by Induced Flows on 2-Simplices

- Suppose we have a confoliation $(\mathbb{M}, \xi, \alpha)$ and a triangulation by 2-simplices such that each 1- and 2-cell of each simplex is transversal to ξ.
- Fixing a simplex σ, there are precisely two vertices p, q of σ such that ξ_p, ξ_q are supporting planes of σ (that is, they do not intersect σ anywhere else.)
- Considering the 1-cell T of σ connecting p and q, there is a diffeomorphism $h_\sigma \colon T \to T$ induced by letting points of T flow along the foliation of σ induced by ξ.

Theorem

- *ξ is a foliation $\implies$ h_σ is the identity.*
- *ξ is a confoliation $\implies$ h_σ is nonstrictly increasing.*
- *ξ is a contact distribution $\implies$ h_σ is strictly increasing.*

Local Models

- Both foliations and contact stuctures are homogenous and admit local normal forms in appropriate local coordinates:
- Foliations can be locally written as

$$dz = 0$$

- Contact structures can be locally written as: (Darboux' Theorem)

$$dz - ydx = 0$$

Local Models

Theorem (Eliashberg and Thurston)

Suppose a 1-form α can be written in local coordinates as

$$\alpha = dz - a(x, y, z)dy$$

Then the associated distribution ξ is:

- *a foliation if and only if $a(x, y, z)$ is independent of y,*
- *a confoliation if and only if $\frac{\partial a}{\partial y} \geq 0$,*
- *a contact structure if and only if $\frac{\partial a}{\partial y} > 0$.*

Deformations of Confoliations

- We say that α can be deformed into a contact structure if there is a smoothly varying family α_t with $\alpha_0 = \alpha$.
- We say that α can be linearly deformed into a contact structure if there is a deformation α_t such that

$$\left.\frac{d(d\alpha_t \wedge \alpha_t)}{dt}\right|_{t=0} > 0$$

- This is equivalent to the existence of a 1-form β such that

$$\langle \alpha, \beta \rangle := \alpha \wedge d\beta + \beta \wedge d\alpha > 0$$

in which case the desired deformation is $\alpha_t = \alpha + t\beta$.

Theorem

If a foliation ξ

1. *has a closed leaf with trivial linear holonomy,*
2. *can be defined by a closed 1-form* $\alpha = d\beta$, *or*
3. *has no holonomy,*

then ξ *cannot be deformed into a contact structure.*

Conversely, if ξ *is a* C^2 *foliation and each of its closed leaves contains a curve with nontrivial holonomy, then* ξ *can be deformed into a contact structure.*

Deformations between foliations

Theorem

Suppose α, β are 1-forms defining foliations on a 3-manifold $\mathbb{M}$, and that $\langle \alpha, \beta \rangle > 0$. Then their associated distributions are transversal, and the 1-form

$$\cos(t)\alpha + \sin(t)\beta$$

is a contact form for $t \in (0, 1)$.