

Comparison Theorems on H-type Foliations

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Riemannian Geometry

Recall that Riemannian geometry is the study of manifolds (generalized Euclidean spaces) that have an infinitesimal notion of distance. In particular, we have the following structures:

- Topological: Notion of “nearness” for points
- Manifold: Local homeomorphisms to \mathbb{R}^n
- Smooth: Derivatives of functions are well-defined
- Riemannian: Distance, and length are well-defined in all directions

Connections

Denoting the space of vector fields $\mathfrak{X}(\mathbb{M})$, an operator

$$\nabla: \mathfrak{X}(\mathbb{M}) \times \mathfrak{X}(\mathbb{M}) \rightarrow \mathfrak{X}(\mathbb{M})$$

such that

- 1 $\nabla_{fX+Y}Z = f\nabla_XZ + \nabla_YZ$
- 2 $\nabla_X(fY) = df(X)Y + f\nabla_XY$

is called a connection.

Curvature

Equipped with a connection, we can define curvature:

- Riemannian Curvature:

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{\nabla_X Y - \nabla_Y X} Z$$

- Sectional Curvature: (for orthonormal X, Y):

$$S(X, Y) = g(R(X, Y)Y, X)$$

- Ricci Curvature:

$$\text{Ric}(X, Y) = \text{Tr}_g(Z \mapsto R(Z, X)Y)$$

- Scalar Curvature:

$$K = \text{Tr}_g(\text{Ric})$$

Levi-Civita Connection

While there are many possible connections on a manifold, there is one in particular that we are interested in.

Theorem

There exists a unique connection ∇ such that

- 1 $\nabla_X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$
- 2 $\nabla_X Y - \nabla_Y X = [X, Y]$

This connection is called the Levi-Civita connection.

Basic Definitions, sub-Riemannian Geometry

Let \mathbb{M} be a smooth manifold. We say that

- $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$ is a sub-Riemannian manifold if
 - \mathcal{H} is a constant rank, bracket generating subbundle of $T\mathbb{M}$,
 - and $g_{\mathcal{H}}$ is a inner product on \mathcal{H} .
- $(\mathbb{M}, \mathcal{H}, g)$ is a sub-Riemannian manifold with metric preserving complement or sRmc-manifold if
 - (\mathbb{M}, g) is a Riemannian manifold,
 - the metric orthogonally splits as $g = g_{\mathcal{H}} \oplus g_{\mathcal{V}}$,
 - and $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$ is a sub-Riemannian manifold.

We denote by \mathcal{V} the orthogonal complement of \mathcal{H} by g .

sub-Riemannian Geometry versus Riemannian Geometry

- Riemannian geometry is distinguished by the Riemannian metric g and thus a definition of distance in all directions.
- sub-Riemannian geometry has a notion of distance, but only in some directions.
- A major goal of sub-Riemannian geometry is to recover Riemannian results (or generalizations).
- The Chow-Rashevskii theorem guarantees that if the horizontal bundle is bracket generating, then any two points are connected by a horizontal path.

Riemannian Comparison Theorem

On a Riemannian manifold assume there exists κ such that

$$\text{Ric}(X, Y) \geq (n-1)\kappa g(X, Y)$$

- Laplacian (Rauch) Comparison Theorem:

$$\Delta r \leq \begin{cases} (n-1)\sqrt{\kappa} \cot(\sqrt{\kappa}r) & \kappa > 0 \\ \frac{n-1}{r} & \kappa = 0 \\ (n-1)\sqrt{|\kappa|} \coth(\sqrt{|\kappa|}r) & \kappa < 0 \end{cases}$$

- Bonnet-Meyers Diameter Estimates: If $\kappa > 0$ then

$$\text{diam}(\mathbb{M}) \leq \frac{\pi}{\sqrt{\kappa}}$$

and the fundamental group of \mathbb{M} must be finite.

Motivating Example: Hopf Fibration

Consider \mathbb{S}^{2n+1} foliated as

$$\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1} \xrightarrow{\pi} \mathbb{C}P^n$$

Defining the horizontal distribution as

$$\mathcal{H} = d\pi^{-1}(T\mathbb{C}P^n)$$

the space $(\mathbb{S}^{2n+1}, \mathcal{H}, g)$ is an sRmc-manifold, foliated with leaves

$$\mathcal{V}_x \cong \mathbb{S}^1$$

Gromov-Hausdorff Convergence

For a sRmc-manifold $(\mathbb{M}, \mathcal{H}, g)$ we define the canonical variation of the metric

$$g_\varepsilon = g_{\mathcal{H}} + \frac{1}{\varepsilon} g_{\mathcal{V}}$$

which in the Gromov-Hausdorff sense

$$(\mathbb{M}, \mathcal{H}, g_\varepsilon) \xrightarrow{\varepsilon \rightarrow 0^+} (\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$$

The key idea of this project is to leverage our knowledge of the Riemannian structure to understand the sub-Riemannian one.

Hladky-Bott Connection

Theorem (Hladky '12 [5])

There exists a unique metric connection ∇ on $(\mathbb{M}, \mathcal{H}, g)$ such that

- ① \mathcal{H} and \mathcal{V} are ∇ -parallel,
- ② The torsion T of ∇ satisfies
 - $T(\mathcal{H}, \mathcal{H}) \subset \mathcal{V}$,
 - $T(\mathcal{V}, \mathcal{V}) \subset \mathcal{H}$
- ③ For every $X, Y \in \Gamma(\mathcal{H}), Z, V \in \Gamma(\mathcal{V})$,
 - $\langle T(X, Z), Y \rangle_{\mathcal{H}} = \langle T(Y, Z), X \rangle_{\mathcal{H}}$
 - $\langle T(Z, X), V \rangle_{\mathcal{V}} = \langle T(V, X), Z \rangle_{\mathcal{V}}$.

This is called the Hladky-Bott connection.

Hladky-Bott Connection

We can explicitly write ∇ in terms of the Levi-Civita connection ∇^g as

$$\nabla_X Y = \begin{cases} \pi_{\mathcal{H}} \nabla_X^g Y & X, Y \in \Gamma(\mathcal{H}) \\ \pi_{\mathcal{H}}[X, Y] + A_X Y & Y \in \Gamma(\mathcal{H}), X \in \Gamma(\mathcal{V}) \\ \pi_{\mathcal{V}}[X, Y] + A_X Y & Y \in \Gamma(\mathcal{V}), X \in \Gamma(\mathcal{H}) \\ \pi_{\mathcal{V}} \nabla_X^g Y & X, Y \in \Gamma(\mathcal{V}) \end{cases}$$

where the tensor A is defined by

$$\langle A_X Y, Z \rangle = \frac{1}{2} ((\mathcal{L}_{X_{\mathcal{V}}} g)(Y_{\mathcal{H}}, Z_{\mathcal{H}}) + (\mathcal{L}_{X_{\mathcal{H}}} g)(Y_{\mathcal{V}}, Z_{\mathcal{V}}))$$

Bundle-like Metrics and Totally Geodesic Foliations

There are two important properties we will require:

- Bundle-like metric: A foliation is said to have a bundle-like metric if the metric locally splits orthogonally. This is equivalent to

$$\mathcal{L}_{\mathcal{V}}g(\mathcal{H}, \mathcal{H}) = 0$$

- Totally geodesic foliation: A foliation is said to be totally geodesic if the geodesics of the fibers are embedded geodesics of the total space. This is equivalent to

$$\mathcal{L}_{\mathcal{H}}g(\mathcal{V}, \mathcal{V}) = 0$$

J Map

On $(\mathbb{M}, \mathcal{H}, g)$ we can associate to each vector field $Z \in \Gamma(TM)$ an endomorphism J_Z of $T\mathbb{M}$ defined by

$$\langle J_Z X, Y \rangle = \langle Z, T(X, Y) \rangle$$

If \mathcal{V} is integrable,

$$\begin{cases} J_Z X \in \mathcal{H} & \text{if } Z \in \mathcal{V}, X \in \mathcal{H} \\ J_Z X = 0 & \text{otherwise} \end{cases}$$

We thus take the perspective

$$J: \mathcal{V} \rightarrow \text{End}(\mathcal{H}), \quad Z \mapsto J_Z$$

H-type Foliations

Definition

Let $(\mathbb{M}, \mathcal{H}, g)$ be a sRmc-manifold. We say that $(\mathbb{M}, \mathcal{H}, g, J)$ is an H-type foliation if

- 1 $(\mathbb{M}, \mathcal{V}, g)$ is a totally geodesic foliation with bundle-like metric,
- 2 \mathcal{V} is integrable, and
- 3 for all $X, Y \in \Gamma(\mathcal{H}), Z \in \Gamma(\mathcal{V})$,

$$\langle J_Z X, J_Z Y \rangle_{\mathcal{H}} = \|Z\|^2 \langle X, Y \rangle_{\mathcal{H}}$$

Parallel Torsion

We also refine the definition of H-type foliations based on the behavior of derivatives of the Hladky-Bott torsion T .

- If $\delta_{\mathcal{H}}T = 0$ we say \mathbb{M} is of Yang-Mills type,
- If $\nabla_{\mathcal{H}}T = 0$ we say \mathbb{M} has horizontally parallel torsion, and
- If $\nabla T = 0$ we say \mathbb{M} has completely parallel torsion.

Lemma

All H-type foliations are Yang-Mills.

Example: Twistor Spaces

Let (\mathbb{M}, g) be a $4n$ -dimensional ($n \geq 2$) quaternionic-Kähler manifold, and fix a quaternionic structure E spanned by $\mathcal{I}, \mathcal{J}, \mathcal{K} \in \text{End}(T\mathbb{M})$. Choosing a metric on E so that $\mathcal{I}, \mathcal{J}, \mathcal{K}$ are orthonormal, we define the twistor space over \mathbb{M} to be the unit sphere bundle of E . In this case, we have

- $\mathcal{H}_x \cong T_x\mathbb{M}$,
- $\mathcal{V}_x \cong \mathbb{C}P^1$.

Here $m = \dim \mathcal{V} = 2$ and there is a quaternionic structure induced by \mathcal{V} acting on \mathcal{H} .

Further Examples of H-type foliations

$\text{rk}(\mathcal{V})$	$\text{rk}(\mathcal{H})$	\mathbb{M}	Torsion
any	2k	Heisenberg Group	C. Parallel
1	2k	Mixed K-contact Sasakian/Hopf fibration	Yang-Mills C. Parallel
2	4k	Salamon Twistor Spaces	H. Parallel
3	4k	Mixed 3K-contact 3-Sasakian/Quaternionic Hopf Torus Bundles over HK	Yang-Mills H. Parallel C. Parallel
4,5,7	8k	Grassmannian Type	H. Parallel
7	8	Octonionic Hopf	H. Parallel

Dimensional Restrictions

For unit $Z \in \mathcal{V}$, the maps J_Z will induce complex, quaternionic, and octonionic structures on \mathcal{H} . As a consequence,

Lemma

Denote $m = \mathbf{rk}(\mathcal{V})$, $n = \mathbf{rk}(\mathcal{H})$. Then

- ① $m \leq n - 1$,
- ② $m = n - 1$ implies $n = 2, 4$, or 8 ,
- ③ $n = 2k$, and furthermore
 - if $m \geq 2$ then $n = 4k$,
 - if $m \geq 4$ then $n = 8k$.

Clifford Structures

Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation with $Z_i, Z_j \in \mathcal{V}$, then

$$J_{Z_i} J_{Z_j} + J_{Z_j} J_{Z_i} = -2\langle Z_i, Z_j \rangle Id_{\mathcal{H}}$$

and so we can extend J in the natural way to

$$J: Cl(\mathcal{V}) \rightarrow \text{End}(\mathcal{H})$$

There is a classification of such Clifford structures over Riemannian manifolds. (A. Moroianu, U. Semmelmann '11 [6])

Parallel Horizontal Clifford Structures

Arising from Riemannian or semi-Riemannian foliations with curvature constancy we have the following notion:

Definition

Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation with horizontally parallel torsion. Then if there exists a map

$$\Psi: \mathcal{V} \times \mathcal{V} \rightarrow Cl_2(\mathcal{V})$$

such that

$$(\nabla_{Z_1} J)_{Z_2} = J_{\Psi(Z_1, Z_2)}$$

for all $Z_1, Z_2 \in \Gamma(\mathcal{V})$ then we say that \mathbb{M} has a parallel horizontal Clifford structure.

This has a strong relationship with the horizontal holonomy of the space.

Parallel Horizontal Clifford Structures

Moreover, these structures are rigid in that they must be of the following form:

Lemma (Baudoin, Grong, Rizzi, & M. '18 [2])

Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation with parallel horizontal Clifford structure. There exists $\kappa \in \mathbb{R}$ such that

$$\Psi(Z_1, Z_2) = -\kappa(Z_1 \cdot Z_2 + \langle Z_1, Z_2 \rangle)$$

for all $Z_1, Z_2 \in \Gamma(\mathcal{V})$; moreover the sectional curvature of the leaves associated to \mathcal{V} is constant and equal to κ^2 .

\mathcal{J}^2 condition

In many model cases, such as the Complex-, Quaternionic-, and Octonionic-Hopf fibrations we have a stronger property:

Definition

Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation. We say that it satisfies the \mathcal{J}^2 condition if for every $Z_1, Z_2 \in \mathcal{V}$ with $\langle Z_1, Z_2 \rangle = 0$ there exists $Z_3 \in \mathcal{V}$ such that

$$J_{Z_1} J_{Z_2} = J_{Z_3}$$

The H-type groups with this property were classified by (M. Cowling, A.H. Dooley, A. Korányi, and F. Ricci '91 [4])

Metric Connections, Jacobi Equation

For the remainder of the talk, let $(M, \mathcal{H}, g_\varepsilon)$

- be a sRmc-manifold,
- equipped with the canonical variation g_ε ,
- having horizontally parallel torsion,
- and satisfying the J^2 condition.

For any $\varepsilon > 0$ the connection

$$\hat{\nabla}_X^\varepsilon Y = \nabla_X Y + J_X^\varepsilon Y$$

will be metric with metric adjoint ∇^ε . For a g_ε -geodesic γ , the Jacobi equation in this setting is

$$\hat{\nabla}_{\dot{\gamma}}^\varepsilon \nabla_{\dot{\gamma}}^\varepsilon W + \hat{R}^\varepsilon(W, \dot{\gamma})\dot{\gamma} = 0$$

The Comparison Principle

Theorem (Baudoin, Grong, Kuwada, & Thalmaier '17 [1])

- Let $x, y \in \mathbb{M}$,
- $\gamma: [0, r_\varepsilon] \rightarrow \mathbb{M}$ a unit speed g_ε -geodesic connecting x, y , and
- W_1, \dots, W_k be a collection of vector fields along γ such that

$$\sum_{i=0}^k \int_0^{r_\varepsilon} \langle \hat{\nabla}_{\dot{\gamma}}^\varepsilon \nabla_{\dot{\gamma}}^\varepsilon W_i + \hat{R}^\varepsilon(W_i, \dot{\gamma})\dot{\gamma}, W_i \rangle_\varepsilon \geq 0$$

then at $y = \gamma(r_\varepsilon)$ it holds that

$$\sum_{i=0}^k \text{Hess}^{\hat{\nabla}^\varepsilon}(r_\varepsilon)(W_i, W_i) \leq \sum_{i=0}^k \langle W_i, \hat{\nabla}_{\dot{\gamma}}^\varepsilon W_i \rangle_\varepsilon$$

with equality if and only if the W_i are Jacobi fields.

Horizontal Splitting

We introduce an orthogonal splitting of the horizontal bundle.
Fixing a vector field $Y \in \mathcal{H}$,

$$\mathcal{H} = \text{span}(Y) \oplus \mathcal{H}_{Riem}(Y) \oplus \mathcal{H}_{Sas}(Y)$$

where

$$\mathcal{H}_{Sas}(Y) = \{J_Z Y \mid Z \in \mathcal{V}\}$$

$$\mathcal{H}_{Riem}(Y) = \{X \in \mathcal{H} \mid X \perp \mathcal{H}_{Sas} \oplus \text{span}(Y)\}$$

Lemma

Denoting $n = \mathbf{rk}(\mathcal{H})$, $m = \mathbf{rk}(\mathcal{V})$, we will have

$$\dim(\mathcal{H}_{Sas}) = m, \quad \dim(\mathcal{H}_{Riem}) = n - m - 1$$

Comparison Functions

Similarly to the Riemannian case, we consider the comparison functions

$$F_{Riem}(r, \kappa) = \begin{cases} \sqrt{\kappa} \cot(\sqrt{\kappa}r) & \text{if } \kappa > 0 \\ \frac{1}{r} & \text{if } \kappa = 0 \\ \sqrt{|\kappa|} \coth(\sqrt{|\kappa|}r) & \text{if } \kappa < 0 \end{cases}$$

$$F_{Sas}(r, \kappa) = \begin{cases} \frac{\sqrt{\kappa}(\sin(\sqrt{\kappa}r) - \sqrt{\kappa}r \cos(\sqrt{\kappa}r))}{2 - 2 \cos(\sqrt{\kappa}r) - \sqrt{\kappa}r \sin(\sqrt{\kappa}r)} & \text{if } \kappa > 0 \\ \frac{4}{r} & \text{if } \kappa = 0 \\ \frac{\sqrt{\kappa}(\sqrt{\kappa}r \cosh(\sqrt{\kappa}r) - \sinh(\sqrt{\kappa}r))}{2 - 2 \cosh(\sqrt{\kappa}r) + \sqrt{\kappa}r \sinh(\sqrt{\kappa}r)} & \text{if } \kappa < 0 \end{cases}$$

These comparison functions will correspond to the splitting of \mathcal{H} .

Hessian Comparisons

Theorem (Baudoin, Grong, Rizzi, & M. '19 [3])

- Let $\gamma: [0, r_\varepsilon] \rightarrow \mathbb{M}$ be a g_ε -geodesic. Then

$$\text{Hess}(r_\varepsilon)(\dot{\gamma}, \dot{\gamma}) \leq \frac{\|\dot{\gamma}\|^2 (1 - \|\dot{\gamma}\|^2)}{r_\varepsilon}$$

- If $\text{Sec}(X \wedge Y) \geq \rho > 0$ for all unit $X, Y \in \mathcal{H}_{\text{Riem}}(\dot{\gamma})$, then

$$\text{Hess}(r_\varepsilon)(X, X) \leq F_{\text{Riem}}(r_\varepsilon, K)$$

- If $\text{Sec}(X \wedge J_Z X) \geq \rho > 0$ for all unit $X \in \mathcal{H}_{\text{Sas}}(\dot{\gamma})$, then

$$\text{Hess}(r_\varepsilon)(X, X) \leq F_{\text{Sas}}(r_\varepsilon, K)$$

Where K is a constant depending on $\rho, \varepsilon, \|\nabla_{\mathcal{V}} r_\varepsilon\|$, and $\|\nabla_{\mathcal{H}} r_\varepsilon\|$.

Horizontal Ricci Curvature

We can define the horizontal Ricci curvature as the trace of the Riemann tensor,

$$\begin{aligned}\operatorname{Ric}_{\mathcal{H}}(X, X) &= \sum_{i=0}^n \langle R^{\nabla}(W_i, X)X, W_i \rangle \\ &= \langle R^{\nabla}(Y, X)X, Y \rangle + \operatorname{Ric}_{Sas}(X, X) + \operatorname{Ric}_{Riem}(X, X)\end{aligned}$$

where the splitting corresponds to the decomposition

$$\mathcal{H} = \operatorname{span}(Y) \oplus \mathcal{H}_{Sas} \oplus \mathcal{H}_{Riem}$$

Diameter Estimates

Theorem (Baudoin, Grong, Rizzi, & M. '19 [3])

Let $\rho > 0$. Then for unit $X \in \mathcal{H}$,

$$\textcircled{1} \quad \frac{\text{Ric}_{\text{Riem}}(X, X)}{n - m - 1} \geq \rho \implies \text{diam}_0(\mathbb{M}) \leq \frac{\pi}{\sqrt{\rho}}$$

$$\textcircled{2} \quad \text{Sec}(X \wedge J_Z X) \geq \rho \implies \text{diam}_0(\mathbb{M}) \leq \frac{2\pi}{\sqrt{\rho}}$$

$$\textcircled{3} \quad \frac{\text{Ric}_{\text{Sas}}(X, X)}{m} \geq \rho \implies \text{diam}_0(\mathbb{M}) \leq \frac{2\pi\sqrt{3}}{\sqrt{\rho}}$$

and in each case the fundamental group of \mathbb{M} must be finite.

The first two of these are sharp, as they are achieved in the complex, quaternionic, and octonionic Hopf fibrations.

sub-Laplacian

Similarly to the horizontal Ricci curvature, we can define the sub-Laplacian as the trace of the Hessian. For the distance function r_ε along a geodesic γ with $Y = \nabla_{\mathcal{H}} r_\varepsilon$,

$$\begin{aligned} \Delta_{\mathcal{H}} r_\varepsilon &= \sum_{i=0}^n \text{Hess}(r_\varepsilon)(W_i, W_i) \\ &= \text{Hess}(r_\varepsilon)(Y, Y) + \sum_{i=0}^m \text{Hess}(r_\varepsilon)(J_{Z_i} Y, J_{Z_i} Y) + \sum_{i=0}^{n-m-1} \text{Hess}(r_\varepsilon)(W_i, W_i) \end{aligned}$$

for appropriate bases $\{W_i\}$ of \mathcal{H} and $\{Z_i\}$ of \mathcal{V} . This splitting corresponds again to the decomposition

$$\mathcal{H} = \text{span}(Y) \oplus \mathcal{H}_{Sas} \oplus \mathcal{H}_{Riem}$$

Laplacian Comparisons

In each component of the horizontal decomposition we can use the previous comparisons on the Hessian to obtain



Theorem (Baudoin, Grong, Rizzi, & M. '19 [3])

Let $(\mathbb{M}, g, \mathcal{H})$ be an H-type foliation with parallel horizontal Clifford structure and satisfying the J^2 condition, and with nonnegative horizontal Bott curvature. Then there exists a $C > 4$ such that

$$\Delta_{\mathcal{H}} r_0 \leq \frac{n - m + 3 + C(m - 1)}{r_0}$$

This is not sharp, but we can recover sharp estimates in each subspace.

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Thank you for your attention!