Comparison Theorems on H-type Foliations

Gianmarco Molino

Fabrice Baudoin, Erlend Grong, and Luca Rizzi

Universitetet i Bergen

18 February, 2020

Riemannian Geometry

Recall that Riemannian geometry is the study of manifolds (generalized Euclidean spaces) that have an infinitesimal notion of distance. In particular, we have the following structures:

- Topological: Notion of “nearness” for points
- Manifold: Local homeomorphisms to $\mathbb{R}^n$
- Smooth: Derivatives of functions are well-defined
- Riemannian: Distance, and length are well-defined in all directions

Connections

Denoting the space of vector fields $\mathfrak{X}(\mathbb{M})$, an operator

$$\nabla\colon \mathfrak{X}(\mathbb{M}) \times \mathfrak{X}(\mathbb{M}) \to \mathfrak{X}(\mathbb{M})$$

such that

1. $\nabla_{fX+Y} Z = f\nabla_X Z + \nabla_Y Z$
2. $\nabla_X (fY) = df(X)Y + f\nabla_X Y$

is called a <u>connection</u>.

Curvature

Equipped with a connection, we can define curvature:

- Riemannian Curvature:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{\nabla_X Y - \nabla_Y X} Z$$

- Sectional Curvature: (for orthonormal X, Y):

$$S(X,Y) = g(R(X,Y)Y, X)$$

- Ricci Curvature:

$$Ric(X,Y) = Tr_g(Z \mapsto R(Z,X)Y)$$

- Scalar Curvature:

$$K = Tr_g(Ric)$$

Levi-Civita Connection

While there are many possible connections on a manifold, there is one in particular that we are interested in.

Theorem

There exists a unique connection ∇ *such that*

1. $\nabla_X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$
2. $T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] = 0$

This connection is called the Levi-Civita connection.

Basic Definitions, sub-Riemannian Geometry

Let $\mathbb{M}$ be a smooth manifold. We say that

- $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$ is a sub-Riemannian manifold if
  - $\mathcal{H}$ is a constant rank, bracket generating subbundle of $T\mathbb{M}$,
  - and $g_{\mathcal{H}}$ is a inner product on $\mathcal{H}$.
- $(\mathbb{M}, \mathcal{H}, g)$ is a sub-Riemannian manifold with metric preserving complement or sRmc-manifold if
  - $(\mathbb{M}, g)$ is a Riemannian manifold,
  - the metric orthogonally splits as $g = g_{\mathcal{H}} \oplus g_{\mathcal{V}}$,
  - and $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$ is a sub-Riemannian manifold.

We denote by $\mathcal{V}$ the orthogonal complement of $\mathcal{H}$ by g.

sub-Riemannian Geometry versus Riemannian Geometry

- Riemannian geometry is distinguished by the Riemannian metric g and thus a definition of distance in all directions.
- sub-Riemannian geometry has a notion of distance, but only in some directions.
- A major goal of sub-Riemannian geometry is to recover Riemannian results (or generalizations).
- The Chow-Rashevskii theorem guarantees that if the horizontal bundle is bracket generating, then any two points are connected by a horizontal path.

Riemannian Comparison Theorem

On a Riemannian manifold assume there exists κ such that

$$Ric(X, Y) \geq (n-1)\kappa g(X, Y)$$

- Laplacian (Rauch) Comparison Theorem:

$$\Delta r \leq \begin{cases} (n-1)\sqrt{\kappa}\cot(\sqrt{\kappa}r) & \kappa > 0 \\ \frac{n-1}{r} & \kappa = 0 \\ (n-1)\sqrt{|\kappa|}\coth(\sqrt{|\kappa|}r) & \kappa < 0 \end{cases}$$

- Bonnet-Meyers Diameter Estimates: If $\kappa > 0$ then

$$\text{diam}(\mathbb{M}) \leq \frac{\pi}{\sqrt{\kappa}}$$

and the fundamental group of $\mathbb{M}$ must be finite.

Motivating Example: Hopf Fibration

Consider $\mathbb{S}^{2n+1}$ foliated as

$$\mathbb{S}^1 \overset{\iota}{\hookrightarrow} \mathbb{S}^{2n+1} \overset{\pi}{\rightarrow} \mathbb{C}P^n$$

We can understand horizontal distribution as

$$\mathcal{H}_x \cong T_{\pi(x)}\mathbb{C}P^n$$

the space $(\mathbb{S}^{2n+1}, \mathcal{H}, g)$ is an sRmc-manifold, foliated with leaves

$$\mathcal{V}_{\iota(x)} \cong T_x\mathbb{S}^1$$

Gromov-Hausdorff Convergence

For a sRmc-manfiold $(\mathbb{M}, \mathcal{H}, g)$ we define the canonical variation of the metric

$$g_\varepsilon = g_\mathcal{H} + \frac{1}{\varepsilon} g_\mathcal{V}$$

which in the Gromov-Hausdorff sense

$$(\mathbb{M}, \mathcal{H}, g_\varepsilon) \xrightarrow{\varepsilon \to 0^+} (\mathbb{M}, \mathcal{H}, g_\mathcal{H})$$

The key idea of this project is to leverage our knowledge of the Riemannian structure to understand the sub-Riemannian one.

Hladky-Bott Connection

Theorem (Hladky '12 [5])

There exists a unique metric connection ∇ on $(\mathbb{M}, \mathcal{H}, g)$ such that

1. $\mathcal{H}$ *and* $\mathcal{V}$ *are* ∇*-parallel,*
2. *The torsion* T *of* ∇ *satisfies*
   - $T(\mathcal{H}, \mathcal{H}) \subset \mathcal{V}$,
   - $T(\mathcal{V}, \mathcal{V}) \subset \mathcal{H}$
3. *For every* $X, Y \in \Gamma(\mathcal{H}), Z, V \in \Gamma(\mathcal{V})$,
   - $\langle T(X, Z), Y\rangle_{\mathcal{H}} = \langle T(Y, Z), X\rangle_{\mathcal{H}}$
   - $\langle T(Z, X), V\rangle_{\mathcal{V}} = \langle T(V, X), Z\rangle_{\mathcal{V}}$.

This is called the Hladky-Bott connection.

Hladky-Bott Connection

We can explicitly write ∇ in terms of the Levi-Civita connection ∇^g as

$$\nabla_X Y = \begin{cases} \pi_{\mathcal{H}} \nabla^g_X Y & X, Y \in \Gamma(\mathcal{H}) \\ \pi_{\mathcal{H}}[X, Y] + A_X Y & Y \in \Gamma(\mathcal{H}), X \in \Gamma(\mathcal{V}) \\ \pi_{\mathcal{V}}[X, Y] + A_X Y & Y \in \Gamma(\mathcal{V}), X \in \Gamma(\mathcal{H}) \\ \pi_{\mathcal{V}} \nabla^g_X Y & X, Y \in \Gamma(\mathcal{V}) \end{cases}$$

where the tensor A is defined by

$$\langle A_X Y, Z \rangle = \frac{1}{2} \left((\mathcal{L}_{X_{\mathcal{V}}} g)(Y_{\mathcal{H}}, Z_{\mathcal{H}}) + (\mathcal{L}_{X_{\mathcal{H}}} g)(Y_{\mathcal{V}}, Z_{\mathcal{V}}) \right)$$

J Map

On $(\mathbb{M}, \mathcal{H}, g)$ we can associate to each vector field $Z \in \Gamma(T\mathbb{M})$ an endomorphism J_Z of $T\mathbb{M}$ defined by

$$\langle J_Z X, Y \rangle = \langle Z, T(X, Y) \rangle$$

If $\mathcal{V}$ is integrable,

$$\begin{cases} J_Z X \in \mathcal{H} & \text{if } Z \in \mathcal{V}, X \in \mathcal{H} \\ J_Z X = 0 & \text{otherwise} \end{cases}$$

We thus take the perspective

$$J \colon \mathcal{V} \to \mathrm{End}(\mathcal{H}), \quad Z \mapsto J_Z$$

Bundle-like Metrics and Totally Geodesic Foliations

There are two important properties we will require:

- Bundle-like metric: A foliation is said to have a bundle-like metric if the metric locally splits orthogonally. This is equivalent to
$$\mathcal{L}_{\mathcal{V}} g(\mathcal{H}, \mathcal{H}) = 0$$
- Totally geodesic foliation: A foliation is said to be totally geodesic if the geodesics of the fibers are embedded geodesics of the total space. This is equivalent to
$$\mathcal{L}_{\mathcal{H}} g(\mathcal{V}, \mathcal{V}) = 0$$

H-type Foliations

Definition

Let $(\mathbb{M}, \mathcal{H}, g)$ be a sRmc-manfiold. We say that $(\mathbb{M}, \mathcal{H}, g, J)$ is an H-type foliation if

1. $(\mathbb{M}, \mathcal{V}, g)$ is a totally geodesic foliation with bundle-like metric,
2. $\mathcal{V}$ is integrable, and
3. for all $X, Y \in \Gamma(\mathcal{H}), Z \in \Gamma(\mathcal{V})$,

$$\langle J_Z X, J_Z Y \rangle_{\mathcal{H}} = \|Z\|^2 \langle X, Y \rangle_{\mathcal{H}}$$

Parallel Torsion

We also refine the definition of H-type foliations based on the behavior of derivatives of the Hladky-Bott torsion T.

- If $\delta_{\mathcal{H}} T = 0$ we say $\mathbb{M}$ is of Yang-Mills type,
- If $\nabla_{\mathcal{H}} T = 0$ we say $\mathbb{M}$ has horizontally parallel torsion, and
- If $\nabla T = 0$ we say $\mathbb{M}$ has completely parallel torsion.

Lemma

All H-type foliations are Yang-Mills.

Example: Twistor Spaces

Let $(\mathbb{M}, g)$ be a $4n$-dimensional $(n \geq 2)$ quaternionic-Kähler manifold, and fix a quaternionic structure E spanned by $\mathcal{I}, \mathcal{J}, \mathcal{K} \in \mathrm{End}(T\mathbb{M})$. Choosing a metric on E so that $\mathcal{I}, \mathcal{J}, \mathcal{K}$ are orthonormal, we define the twistor space over $\mathbb{M}$ to be the unit sphere bundle of E. In this case, we have

- $\mathcal{H}_x \cong T_x\mathbb{M}$,
- $\mathcal{V}_x \cong \mathbb{C}P^1$.

Here $m = \dim \mathcal{V} = 2$ and there is a quaternionic structure induced by $\mathcal{V}$ acting on $\mathcal{H}$.

Further Examples of H-type foliations

rk($\mathcal{V}$)	**rk**($\mathcal{H}$)	$\mathbb{M}$	Torsion
any	2k	H-type Group	C. Parallel
1	2k	Mixed K-contact	Yang-Mills
		Sasakian/Hopf fibration	C. Parallel
2	4k	Salamon Twistor Spaces	H. Parallel
3	4k	Mixed 3K-contact	Yang-Mills
		3-Sasakian/Quaternionic Hopf	H. Parallel
		Torus Bundles over HK	C. Parallel
4,5,7	8k	Grassmannian Type	H. Parallel
7	8	Octonionic Hopf	H. Parallel

Dimensional Restrictions

For unit $Z \in \mathcal{V}$, the maps J_Z will induce complex, quaternionic, and octonionic structures on $\mathcal{H}$. As a consequence,

Lemma

Denote $m = \mathbf{rk}(\mathcal{V}), n = \mathbf{rk}(\mathcal{H})$. Then

1. $m \leq n - 1$,
2. $m = n - 1$ *implies* $n = 2, 4,$ *or* 8,
3. $n = 2k$, *and furthermore*
   - *if* $m \geq 2$ *then* $n = 4k$,
   - *if* $m \geq 4$ *then* $n = 8k$.

Clifford Structures

Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation with $Z_i, Z_j \in \mathcal{V}$, then

$$J_{Z_i} J_{Z_j} + J_{Z_j} J_{Z_i} = -2\langle Z_i, Z_j \rangle Id_{\mathcal{H}}$$

and so we can extend J in the natural way to

$$J \colon Cl(\mathcal{V}) \to \mathrm{End}(\mathcal{H})$$

There is a classification of such Clifford structures over Riemannian manifolds. (A. Moroianu, U. Semmelmann '11 [6])

Parallel Horizontal Clifford Structures

Arising from Riemannian or semi-Riemannian foliations with curvature constancy we have the following notion:

Definition

Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation with horizontally parallel torsion. Then if there exists a map

$$\Psi \colon \mathcal{V} \times \mathcal{V} \to Cl_2(\mathcal{V})$$

such that

$$(\nabla_{Z_1} J)_{Z_2} = J_{\Psi(Z_1, Z_2)}$$

for all $Z_1, Z_2 \in \Gamma(\mathcal{V})$ then we say that $\mathbb{M}$ has a parallel horizontal Clifford structure.

This has a strong relationship with the horizontal holonomy of the space.

Parallel Horizontal Clifford Structures

Moreover, these structures are rigid in that they must be of the following form:

Lemma (Baudoin, Grong, Rizzi, & M. '18 [2])

Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation with parallel horizontal Clifford structure. There exists $\kappa \in \mathbb{R}$ such that

$$\Psi(Z_1, Z_2) = -\kappa(Z_1 \cdot Z_2 + \langle Z_1, Z_2 \rangle)$$

for all $Z_1, Z_2 \in \Gamma(\mathcal{V})$; moreover the sectional curvature of the leaves associated to $\mathcal{V}$ is constant and equal to κ^2.

J^2 condition

In many model cases, such as the Complex-, Quaternionic-, and Octonionic-Hopf fibrations we have a stronger property:

Definition

Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation. We say that it satisfies the J^2 condition if for every $Z_1, Z_2 \in \mathcal{V}$ with $\langle Z_1, Z_2 \rangle = 0$ there exists $Z_3 \in \mathcal{V}$ such that

$$J_{Z_1} J_{Z_2} = J_{Z_3}$$

The H-type groups with this property were classified by (M. Cowling, A.H. Dooley, A. Korányi, and F.Ricci '91 [4])

Metric Connections, Jacobi Equation

For the remainder of the talk, let $(M, \mathcal{H}, g_\varepsilon)$

- be a sRmc-manifold,
- equipped with the canonical variation g_ε,
- having horizontally parallel torsion,
- and satisfying the J^2 condition.

For any $\varepsilon > 0$ the connection

$$\hat{\nabla}^\varepsilon_X Y = \nabla_X Y + J^\varepsilon_X Y$$

will be metric with metric adjoint ∇^ε. For a g_ε-geodesic γ, the Jacobi equation in this setting is

$$\hat{\nabla}^\varepsilon_{\dot{\gamma}} \nabla^\varepsilon_{\dot{\gamma}} W + \hat{R}^\varepsilon(W, \dot{\gamma})\dot{\gamma} = 0$$

The Comparison Principle

Theorem (Baudoin, Grong, Kuwada, & Thalmaier '17 [1])

- *Let* $x, y \in \mathbb{M}$,
- $\gamma\colon [0, r_\varepsilon] \to \mathbb{M}$ *a unit speed* g_ε*-geodesic connecting* x, y, *and*
- $W_1, \cdots, W_k$ *be a collection of vector fields along* γ *such that*

$$\sum_{i=0}^{k} \int_0^{r_\varepsilon} \langle \hat{\nabla}^\varepsilon_{\dot\gamma} \nabla^\varepsilon_{\dot\gamma} W_i + \hat{R}^\varepsilon(W_i, \dot\gamma)\dot\gamma, W_i \rangle_\varepsilon \geq 0$$

then at $y = \gamma(r_\varepsilon)$ *it holds that*

$$\sum_{i=0}^{k} \mathrm{Hess}^{\hat{\nabla}^\varepsilon}(r_\varepsilon)(W_i, W_i) \leq \sum_{i=0}^{k} \langle W_i, \hat{\nabla}^\varepsilon_{\dot\gamma} W_i \rangle_\varepsilon$$

with equality if and only if the W_i *are Jacobi fields.*

Horizontal Splitting

We introduce an orthogonal splitting of the horizontal bundle. Fixing a vector field $Y \in \mathcal{H}$,

$$\mathcal{H} = \mathrm{span}(Y) \oplus \mathcal{H}_{Riem}(Y) \oplus \mathcal{H}_{Sas}(Y)$$

where

$$\mathcal{H}_{Sas}(Y) = \{J_Z Y | Z \in \mathcal{V}\}$$

$$\mathcal{H}_{Riem}(Y) = \{X \in \mathcal{H} | X \perp \mathcal{H}_{Sas} \oplus \mathrm{span}(Y)\}$$

Lemma

Denoting $n = \mathbf{rk}(\mathcal{H}), m = \mathbf{rk}(\mathcal{V})$, *we will have*

$$\dim(\mathcal{H}_{Sas}) = m, \quad \dim(\mathcal{H}_{Riem}) = n - m - 1$$

Comparison Functions

Simliarly to the Riemannian case, we consider the comparision functions

$$F_{Riem}(r,\kappa) = \begin{cases} \sqrt{\kappa}\cot(\sqrt{\kappa}r) & \text{if } \kappa > 0 \\ \frac{1}{r} & \text{if } \kappa = 0 \\ \sqrt{|\kappa|}\coth(\sqrt{|\kappa|}r) & \text{if } \kappa < 0 \end{cases}$$

$$F_{Sas}(r,\kappa) = \begin{cases} \frac{\sqrt{\kappa}\left(\sin(\sqrt{\kappa}r)-\sqrt{\kappa}r\cos(\sqrt{\kappa}r)\right)}{2-2\cos(\sqrt{\kappa}r)-\sqrt{\kappa}r\sin(\sqrt{\kappa}r)} & \text{if } \kappa > 0 \\ \frac{4}{r} & \text{if } \kappa = 0 \\ \frac{\sqrt{\kappa}\left(\sqrt{\kappa}r\cosh(\sqrt{\kappa}r)-\sinh(\sqrt{\kappa}r)\right)}{2-2\cosh(\sqrt{\kappa}r)+\sqrt{\kappa}r\sinh(\sqrt{\kappa}r)} & \text{if } \kappa < 0 \end{cases}$$

These comparison functions will correspond to the splitting of $\mathcal{H}$.

Hessian Comparisons

Theorem (Baudoin, Grong, Rizzi, & M. '19 [3])

- *Let* $\gamma\colon [0, r_\varepsilon] \to \mathbb{M}$ *be a* g_ε*-geodesic. Then*

$$\mathrm{Hess}(r_\varepsilon)(\dot\gamma, \dot\gamma) \le \frac{\|\dot\gamma\|^2 \left(1 - \|\dot\gamma\|^2\right)}{r_\varepsilon}$$

- *If* $\mathrm{Sec}(X \wedge Y) \ge \rho > 0$ *for all unit* $X, Y \in \mathcal{H}_{Riem}(\dot\gamma)$*, then*

$$\mathrm{Hess}(r_\varepsilon)(X, X) \le F_{Riem}(r_\varepsilon, K)$$

- *If* $\mathrm{Sec}(X \wedge J_Z X) \ge \rho > 0$ *for all unit* $X \in \mathcal{H}_{Sas}(\dot\gamma)$*, then*

$$\mathrm{Hess}(r_\varepsilon)(X, X) \le F_{Sas}(r_\varepsilon, K)$$

Where K *is a constant depending on* $\rho, \varepsilon, \|\nabla_{\mathcal{V}} r_\varepsilon\|$*, and* $\|\nabla_{\mathcal{H}} r_\varepsilon\|$*.*

Horizontal Ricci Curvature

We can define the horizontal Ricci curvature as the trace of the Riemann tensor,

$$\begin{aligned} \mathrm{Ric}_{\mathcal{H}}(X,X) &= \sum_{i=0}^{n} \langle R^{\nabla}(W_i,X)X, W_i\rangle \\ &= \langle R^{\nabla}(Y,X)X, Y\rangle + \mathrm{Ric}_{Sas}(X,X) + \mathrm{Ric}_{Riem}(X,X) \end{aligned}$$

where the splitting corresponds to the decomposition

$$\mathcal{H} = \mathrm{span}(Y) \oplus \mathcal{H}_{Sas} \oplus \mathcal{H}_{Riem}$$

Diameter Estimates

Theorem (Baudoin, Grong, Rizzi, & M. '19 [3])

Let $\rho > 0$. Then for unit $X \in \mathcal{H}$,

1. $\dfrac{\mathrm{Ric}_{Riem}(X,X)}{n-m-1} \geq \rho \implies \mathrm{diam}_0(\mathbb{M}) \leq \dfrac{\pi}{\sqrt{\rho}}$
2. $\mathrm{Sec}(X \wedge J_Z X) \geq \rho \implies \mathrm{diam}_0(\mathbb{M}) \leq \dfrac{2\pi}{\sqrt{\rho}}$
3. $\dfrac{\mathrm{Ric}_{Sas}(X,X)}{m} \geq \rho \implies \mathrm{diam}_0(\mathbb{M}) \leq \dfrac{2\pi\sqrt{3}}{\sqrt{\rho}}$

and in each case the fundamental group of $\mathbb{M}$ must be finite.

The first two of these are sharp, as they are achieved in the complex, quaternionic, and octonionic Hopf fibrations.

sub-Laplacian

Similarly to the horizontal Ricci curvature, we can define the sub-Laplacian as the trace of the Hessian. For the distance function r_ε along a geodesic γ with $Y = \nabla_{\mathcal{H}} r_\varepsilon$,

$$\Delta_{\mathcal{H}} r_\varepsilon = \sum_{i=0}^{n} \mathrm{Hess}(r_\varepsilon)(W_i, W_i)$$

$$= \mathrm{Hess}(r_\varepsilon)(Y, Y) + \sum_{i=0}^{m} \mathrm{Hess}(r_\varepsilon)(J_{Z_i}Y, J_{Z_i}Y) + \sum_{i=0}^{n-m-1} \mathrm{Hess}(r_\varepsilon)(W_i, W_i)$$

for appropriate bases $\{W_i\}$ of $\mathcal{H}$ and $\{Z_i\}$ of $\mathcal{V}$. This splitting corresponds again to the decomposition

$$\mathcal{H} = \mathrm{span}(Y) \oplus \mathcal{H}_{Sas} \oplus \mathcal{H}_{Riem}$$

Laplacian Comparisons

In each component of the horizontal decomposition we can use the previous comparisons on the Hessian to obtain

Theorem (Baudoin, Grong, Rizzi, & M. '19 [3])

Let $(\mathbb{M}, g, \mathcal{H})$ be an H-type foliation with parallel horizontal Clifford structure and satisfying the J^2 condition, and with nonnegative horizontal Bott curvature. Then there exists a $C > 4$ such that

$$\Delta_{\mathcal{H}} r_0 \leq \frac{n - m + 3 + C(m-1)}{r_0}$$

This is not sharp, but we can recover sharp estimates in each subspace.

References I

- F. Baudoin, E. Grong, K. Kuwada, and A. Thalmaier.
  Sub-Laplacian comparison theorems on totally geodesic Riemannian foliations.
  arXiv e-prints, Jun 2017, 1706.08489.

- F. Baudoin, E. Grong, G. Molino, and L. Rizzi.
  H-type Foliations.
  arXiv e-prints, Dec 2018, 1812.02563.

- F. Baudoin, E. Grong, G. Molino, and L. Rizzi.
  Comparison theorems on H-type sub-Riemannian manifolds.
  arXiv e-prints, Sept 2019, 1909.03532.

- M. Cowling, A. Dooley, A. Korányi, and F. Ricci.
  H-type groups and Iwasawa decompositions.
  Adv. Math., 81(1):1–41, 1991.

References II

R. K. Hladky.
Connections and curvature in sub-Riemannian geometry.
Houston J. Math., 38(4):1107–1134, 2012.

A. Moroianu and U. Semmelmann.
Clifford structures on Riemannian manifolds.
Adv. Math., 228(2):940–967, 2011.

Thank you for your attention!