Heat Kernel Methods in Index Theory

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What is Index Theory?

Generally, index theory aims to identify interesting invariants (usually associated to pseudodifferential operators) and to determine formulas for their computation that reveal nontrivial properties of the invariants; these have numerous applications, such as

- Existence of metrics on spaces with known curvature bounds
- Proof of Gromov-Lawson, K-theory,
- Positive Mass theorems,
- Lefschetz Fixed-Point formula

Some History

- Gauss-Bonnet Theorem (1848) computed the alternating sum of the number of vertices, edges, and faces of a triangulation of a surface by the integral of the Gauss curvature of the surface. This was extended by Chern-Gauss-Bonnet to manifolds of all dimensions.
- Riemann-Roch (1865) relates the dimension of meromorphic functions on a Riemann surface with the topological genus.
- Atiyah-Singer (1963) generalized earlier index theorems, relating an index associated to Dirac operators to the integral of K-theoretic topological quantities.
- Atiyah-Bott-Patodi (1973) introduce the heat kernel approach.

Motivating Theorem

Theorem (Chern-Gauss-Bonnet Theorem)

Suppose $\mathbb{M}$ is a $d = 2l$ dimensional, compact, oriented Riemannian manifold. Then

$$\chi(\mathbb{M}) = \int_{\mathbb{M}} e(x)\ dx$$

where $\chi(\mathbb{M})$ denotes the Euler characteristic,

$$e(x) = \frac{\mathrm{Str}(\Omega)^l}{(4\pi)^l l!}$$

is the Euler form and Ω denotes the curvature form associated to the Riemannian curvature tensor of $\mathbb{M}$.

Exterior Derivative and Differential Forms

On a smooth manifold, we can consider the exterior algebra of the cotangent bundle

$$\Lambda^*\mathbb{M} = \bigoplus_{0\leq p\leq d} \Lambda^p\mathbb{M}$$

that is the tensor algebra of the cotangent bundle modulo alternating products

$$x \wedge y = -y \wedge x$$

and the exterior derivative $d \colon \Lambda^p\mathbb{M} \to \Lambda^{p+1}\mathbb{M}$ defined as the unique operator for which

1. For smooth functions f, df is the differential,
2. $d^2 = 0$, that is $d(d\alpha) = 0$ for all α, and
3. $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p\alpha \wedge d\beta$ where $\beta \in \Lambda^p\mathbb{M}$.

Euler Characteristic

We first recall the definition of the Euler characteristic:

$$\chi(\mathbb{M}) = \sum_{p=0}^{d} (-1)^p \dim H^p_{dR}(\mathbb{M})$$

Where

$$H^p_{dR}(\mathbb{M}) = \frac{\ker\{d^p \colon \Lambda^p\mathbb{M} \to \Lambda^{p+1}\mathbb{M}\}}{\operatorname{im}\{d^{p-1} \colon \Lambda^{p-1}\mathbb{M} \to \Lambda^p\mathbb{M}\}}$$

denotes the p-th deRham cohomology group.

Hodge-deRham Laplacian

Recall the Hodge-deRham Laplacian on differential forms

$$\Box_{\mathbb{M}} := d\delta + \delta d$$

where d denotes the exterior derivative and δ denotes its formal adjoint under the inner product induced by the Riemannian metric.
That is for $\alpha \in \Lambda^{p-1}\mathbb{M}, \beta \in \Lambda^p\mathbb{M}$,

$$\langle d\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle$$

for the inner product on forms $\alpha, \beta \in \Lambda^p\mathbb{M}$

$$\langle \alpha, \beta \rangle = \int_{\mathbb{M}} \alpha \wedge *_g \beta \; dvol$$

Hodge Decomposition and Isomorphism

There is a decomposition of the space of smooth p-forms

$$\Lambda^p\mathbb{M} = \ker \square^p_{\mathbb{M}} \oplus \operatorname{im} d^{p-1} \oplus \operatorname{im} \delta^{p+1}$$

From the decomposition it follows

Theorem (Hodge Isomorphism)

$$\ker \square^p_{\mathbb{M}} \cong H^p_{dR}(\mathbb{M})$$

Heat Equation

It is known that there exists a heat kernel on differential forms

$$p^*(t,x,y)\colon \Lambda_y^*\mathbb{M} \to \Lambda_x^*\mathbb{M}$$

That is, if $\theta(t,x) \in \Lambda_x^*\mathbb{M}$ is a solution to the heat equation

$$\begin{cases} \frac{\partial \theta}{\partial t} = -\frac{1}{2}\,\square_{\mathbb{M}}\theta \\ \theta(0,x) = \theta_0(x) \end{cases}$$

then

$$e^{-t\,\square_{\mathbb{M}}}\theta_0(x) := \theta(t,x) = \int_{\mathbb{M}} p^*(t,x,y)\theta(y)\;dy$$

Eigenvalue Expansion

The elliptic operator $-\square_{\mathbb{M}}$ has discrete spectrum

$$\lambda_0 = 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$$

and the heat kernel can be expanded as

$$p^*(t, x, y) = \sum_{i=0}^{\infty} e^{-\lambda_i t/2} f_n(x) \otimes f_n^*(y)$$

for orthonormal eigenforms f_n.

Eigenspace Decomposition

Let

$$\mu_0 = 0 < \mu_1 < \mu_2 < \cdots$$

be the distinct eigenvalues of $-\Box_{\mathbb{M}}$ and E_i^p be the space of p-forms in the μ_i-eigenspace. For $i \geq 1 \implies \mu_i > 0$ the Hodge Decomposition reads

$$E_i^p = dE_i^{p-1} \oplus \delta E_i^{p+1}$$

and we can conclude that

$$\sum_{p=0}^{d} (-1)^p \dim E_i^p = 0$$

This crucial observation was first made by Mckean and Singer.

Supertrace Formula

Defining the supertrace of a linear endomorphism T of an n-dimensional vector space V by

$$\text{Str } T = \sum_{p=0}^{n} (-1)^p Tr_{\Lambda^p V} T$$

We have that

Theorem

For any $t > 0$,

$$\chi(\mathbb{M}) = \int_{\mathbb{M}} \text{Str } p^*(t, x, x) \; dx$$

Proof of Supertrace Formula

Applying the Hodge Isomorphism and then the Eigenspace Decomposition,

$$\begin{aligned}
\chi(\mathbb{M}) &= \sum_{p=0}^{d} (-1)^p \dim H^p_{dR}(\mathbb{M}) \\
&= \sum_{p=0}^{d} (-1)^p \dim \ker \Box^p_{\mathbb{M}} := \sum_{p=0}^{d} (-1)^p \dim E^p_0 \\
&= \sum_{i=0}^{\infty} e^{-\lambda_i t/2} \sum_{p=0}^{d} (-1)^p \dim E^p_i \\
&= \int_{\mathbb{M}} \mathrm{Str}\, p^*(t,x,x)\ dx
\end{aligned}$$

Weitzenböck Formula

It remains to compute $\operatorname{Str} p^*(t, x, x)$. Importantly, observe that our expression must be independent of t.

There is a famous relation between the Hodge-deRham Laplacian $\Box_{\mathbb{M}}$ and the Bochner Laplacian $\nabla^*\nabla$ (for the Levi-Civita connection ∇ with associated Riemannian curvature tensor R^{∇})

Theorem (Weitenböck Decomposition)

$$\Box_{\mathbb{M}} = -\nabla^*\nabla + R^{\nabla}$$

Brownian Motion and Feynman-Kac

It is standard to interpret the heat flow generated by the Bochner Laplacian as a Brownian motion in the sense that

$$e^{-t\nabla^*\nabla} f(X_0) = \mathbb{E}(f(X_t))$$

where $\mathbb{E}$ is the expectation with respect to a Weiner measure. Applying the Feynman-Kac formula with the Weitenböck decomposition, we see that

$$e^{-t\Box_{\mathbb{M}}}\theta_0(x) = \mathbb{E}(M_t U_t^{-1}\theta_0(X_t))$$

where U_t^{-1} is the stochastic parallel transport along a Brownian bridge and M_t satisfies

$$\frac{dM_t}{dt} = \frac{1}{2} M_t \Omega(U_t), \quad M_0 = Id$$

Short-time Asymptotic

As a consequence of this interpretation, we can write

$$p^*(t,x,x) = p(t,x,x)\mathbb{E}_{x,x;t}(M_t U_t^{-1})$$

where $p(t,x,y)$ is the heat kernel on functions.
Explicitly computing

$$e(x) = \lim_{t\to 0^+} \operatorname{Str} p^*(t,x,x)$$

will complete the proof of Chern-Gauss-Bonnet.

Asymptotic behavior

Generally it is known that

$$p(t,x,x) \sim \frac{1}{(2\pi t)^{d/2}} \quad \text{as } t \to 0^+$$

thus it is necessary that

$$\operatorname{Str} M_t U_t^{-1} \sim C t^{d/2} \quad \text{as } t \to 0^+$$

Approaches to computing $\lim_{t\to 0^+} \operatorname{Str} M_t U_t^{-1}$

There are numerous approaches;

- Malliavin calculus, due to Bismut
- Parametrix, see Hsu or Rosenberg,
- Brownian-Chen, see Baudoin

Brownian-Chen Series

Let's consider $\mathbb{R}[X_0, \ldots, X_d]$, the noncommutative algebra over $\mathbb{R}$ of formal series of the form

$$Y = \sum_{k \geq 0} \sum_{i_1 \cdots i_k} a_{i_1 \cdots i_k} X_{i_1} \cdots X_{i_k}$$

equipped with bracket

$$[X, Y] = XY - YX$$

and for the word $I = (i_1, \ldots, i_k)$ the commutator

$$X_I = [X_{i_1}, [X_{i_2}, \ldots, [X_{i_{k-1}}, X_{i_k}] \ldots]$$

Brownian-Chen Series

Defining

$$\Lambda_I(B)_t = \sum_{\sigma \in \mathcal{S}_k} \frac{(-1)^{e(\sigma)}}{k^2 \binom{k-1}{e(\sigma)}} \int_{\Delta^k[0,t]} \circ dB^{\sigma^{-1}(I)}$$

we have the result from Rough Paths theory

Lemma

$$\exp\left(t\left(X_0 + \frac{1}{2}\sum_{i=1}^{d} X_i^2\right)\right) = \mathbb{E}\exp\left(\sum_{k\geq 1}\sum_{I\in\{0,\dots,d\}^k} \Lambda_I(B)_t X_I\right)$$

Local Expansion of the Laplacian

Since the Hodge-deRham Laplacian $\square_{\mathbb{M}}$ can be written at the center of a local frame $\{e_i\}_{1\leq i\leq d}$ as

$$\square_{\mathbb{M}} = R^{\nabla} - \nabla^*\nabla = R^{\nabla} + \frac{1}{2}\sum_{i=1}^{d} \nabla_{e_i}^2,$$

we can find the short-time asymptotic

Lemma

For $N \geq 1$ as $t \to 0^+$,

$$p^*(t,x,x) = d_t^N(x)\mathbb{E}\left(\exp\left(\sum_{I,d(I)\leq N} \Lambda_I(B)_t(\nabla_I - \nabla_{V_I})(x)\right)\middle| \sum_{I,d(I)\leq N} \Lambda_I(B)_t V_I(x) = 0\right) + O(t^{\frac{N-1+d}{2}})$$

Fantastic Cancellation

Because $\nabla_I - \nabla_{V_I} = c(\mathcal{F}_I)$ for some $\mathcal{F}_I$ in the even exterior algebra of the cotangent bundle,

$$\mathrm{Str}\left(\sum_{I,|I|\leq d, 0\notin I} t^{|I|/2}\Lambda_I(B)_t(\nabla_I - \nabla_{V_I})(x)I\right)^k = 0$$

whenever $k < d/2$. Thus we have

Lemma

$$\mathrm{Str}\left(\exp\left(\sum_{I,|I|\leq d, 0\notin I} t^{|I|/2}\Lambda_I(B)_t(\nabla_I - \nabla_{V_I})(x)I\right)\right)$$

$$= \frac{1}{(d/2)!}\mathrm{Str}\left(\sum_{I,|I|\leq d, 0\notin I} t^{|I|/2}\Lambda_I(B)_t(\nabla_I - \nabla_{V_I})(x)I\right)^{d/2}$$

Conclusion of the Proof

As a consequence of the short-time asymptotic and the fantastic cancellation,

Theorem (Local Chern-Gauss-Bonnet)

$$e(x) := \lim_{t\to 0^+} \operatorname{Str} p^*(t,x,x) = \begin{cases} \frac{\operatorname{Str}(\Omega)^l}{(4\pi)^l l!} & d = 2l \\ 0 & d \textit{ is odd} \end{cases}$$

which immediately gives us the Chern-Gauss-Bonnet Theorem considering the supertrace formula.

Atiyah-Singer

This idea and the heat kernel method can be generalized; one of the most celebrated results is the

Theorem (Atiyah-Singer Index)

Let D be a Dirac operator on the twisted spin bundle $G^{\pm} = \mathcal{S}(\mathbb{M})^{\pm} \otimes \xi$ of a spin manifold $\mathbb{M}$. Then

$$\dim \ker D^{+} - \dim \ker D^{-} = \int_{\mathbb{M}} \hat{A}(TM) \wedge ch(\xi)$$

Relation to Chern-Gauss-Bonnet

In the Chern-Gauss-Bonnet setting we can let $G^{\pm}$ be the space of even and odd forms $\Lambda^*\mathbb{M}$ in which case the Dirac operator will be

$$D = d + \delta$$

It can be quickly verified that

$$\chi(\mathbb{M}) = \dim\ker D^+ - \dim\ker D^-$$
$$\int_{\mathbb{M}} e(x)\ dx = \int_{\mathbb{M}} \hat{A}(TM) \wedge ch(\xi)$$

Thus, the Atiyah-Singer Index Theorem is a strict generalization.

Indication of Heat Kernel Approach

Since in the previous example

$$\Box_{\mathbb{M}} = d\delta + \delta d = (d+\delta)^2 = D^2$$

we are motivated to approach the proof by considering the heat kernel $p^D(t,x,y)$ for the operator e^{-tD^2}. In fact, the supertrace formula

$$\dim\ker D^+ - \dim\ker D^- = \int_{\mathbb{M}} \operatorname{Str} p^D(t,x,x)\ dx$$

follows very similarly as for Chern-Gauss-Bonnet.

Lichnerowicz Formula

In this setting we have the Weitzenböck decomposition

Theorem (Lichnerowicz)

$$D^2 = -\nabla^*\nabla + \frac{S}{4} + \frac{1}{2}\sum_{j,k=1}^{d} c(X_j)c(X_k) \otimes L(X_j, X_k)$$

Here S denotes the scalar curvature, c is the Clifford multiplication on the spin bundle, L is the curvature operator on ξ, and $\{X_j\}$ is an orthonormal frame of $T\mathbb{M}$.

Local Atiyah-Singer

- There is a local Atiyah-Singer theorem that is proved by considering the heat equation

$$\begin{cases} \frac{\partial f}{\partial t} = -\frac{1}{2} D^2 f \\ f(0,x) = f_0(x) \end{cases}$$

- This can be solved using parametrix, introducing an appropriate local expression for the heat kernel via the Feynman-Kac formula and the Lichnerowicz formula, but becomes increasingly difficult.
- Importantly, the map U_t^{-1} is no longer an isometry; for this reason heat parametrix approaches become significantly untenable, but the Brownian-Chen approach remains viable.

Local Atiyah-Singer

The idea of "fantastic" cancellation still holds since the equivalent $\mathcal{F}_I$ belong to the even Clifford algebra of the cotangent bundle, and a computation will give

Theorem

$$\lim_{t \to 0^+} \operatorname{Str} p^D(t, x, x)\ dx \approx \hat{A}(T\mathbb{M}) \wedge ch(\xi)$$

sub-Riemannian Geometry

- A sub-Riemannian manifold is a smooth manifold equipped with a subbundle $\mathcal{H}$ of the tangent bundle such that there is an inner product g_x defined only on $\mathcal{H}$.
- In this setting δ is not defined on the entire tangent bundle, and as a consequence it is not possible to define a Laplacian as the square of a Dirac operator $D = d + \delta$. However, a horizontal Laplacian $\Delta_{\mathcal{H},\epsilon}$ for $\epsilon \in (0, +\infty]$ is still sensible.
- McKean-Singer type supertrace results can be recovered.

Further Reading

- Chen series and Atiyah-Singer theorem, F. Baudoin
- The index of elliptic operators, M.F. Atiyah, I.M. Singer
- The Atiyah-Singer Thoerems: A Probabilistic Approach, J.M. Bismut
- Heat Kernels and Dirac Operators, N. Berline, E. Getzler, M. Vergne