Sub-Laplacian Comparison Theorems on H-Type Foliations

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Riemannian Geometry

- Riemannian manifolds allow for many notions of curvature
- Metric, analytic, and even topological properties can be determined from a knowledge of curvature
- How do these ideas fit in a subRiemannian setting?

Consequences of Laplacian Comparisons

Let $(\mathbb{M}, g)$ be a Riemannian manifold of dimension m and suppose there exists $\kappa \in \mathbb{R}$ such that $Ric \geq (n-1)\kappa g$.

Theorem (Bonnet-Meyers)

If $\kappa > 0$ *then*

- $\mathrm{diam}(\mathbb{M}) \leq \frac{\pi}{\sqrt{\kappa}}$
- *The fundamental group of* $\mathbb{M}$ *must be finite.*

Consequences of Laplacian Comparisons

Let $(\mathbb{M}, g)$ be a Riemannian manifold of dimension m and suppose there exists $\kappa \in \mathbb{R}$ such that $Ric \geq (n-1)\kappa g$.

Theorem (Bishop-Gromov)

Let $\mathbb{M}_\kappa^m$ be the Riemannian manifold of dimension m and constant sectional curvature κ. Denote by $B_{\mathbb{M}}(p, r)$ the Riemannian ball of radius r around $p \in \mathbb{M}$. Then

$$\phi(r) = \frac{B_{\mathbb{M}}(p, r)}{B_{\mathbb{M}_\kappa^m}(p_\kappa, r)}$$

is nonincreasing on $(0, \infty)$.

Connections

Denoting the space of vector fields $\Gamma(T\mathbb{M})$, an operator

$$\nabla : \Gamma(T\mathbb{M}) \times \Gamma(T\mathbb{M}) \to \Gamma(T\mathbb{M})$$

such that

1. $\nabla_{fX+Y} Z = f\nabla_X Z + \nabla_Y Z$
2. $\nabla_X (fY) = df(X)Y + f\nabla_X Y$

is called a connection.

Levi-Civita Connection

Riemannian Geometry is characterized by the Levi-Civita connection,

Theorem

Let $(\mathbb{M}, g)$ be a Riemannian manifold. There exists a unique connection ∇ on $\mathbb{M}$ such that

1. $\nabla_X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$
2. $\nabla_X Y - \nabla_Y X = [X, Y]$

Curvature

- Riemannian Curvature:

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{\nabla_X Y - \nabla_Y X} Z$$

- Sectional Curvature: (for orthonormal X, Y):

$$K(X, Y) = g(R(X, Y)Y, X)$$

- Ricci Curvature:

$$Ric(X, Y) = Tr(Z \mapsto g(R(Z, X)Y, Z))$$

- Scalar Curvature:

$$s(X) = Tr(Y \mapsto Ric(X, Y))$$

Some Definitions

We set some notation. For a Riemannian manifold $(\mathbb{M}, g)$ and a point $p \in \mathbb{M}$, we define the distance function

$$d_p \colon \mathbb{M} \to \mathbb{R}, \;\; d_p(q) = d(p, q)$$

Let $\gamma \colon [0, L] \to M$ be a minimizing geodesic. Then we define the curvatures

$$K^+(t) = \sup\{K(X_{\gamma(t)}, Y_{\gamma(t)}) \colon \gamma'(t) \in \mathrm{Span}(X_{\gamma(t)}, Y_{\gamma(t)})\}$$
$$K^-(t) = \inf\{K(X_{\gamma(t)}, Y_{\gamma(t)}) \colon \gamma'(t) \in \mathrm{Span}(X_{\gamma(t)}, Y_{\gamma(t)})\}$$

We define

$$\mathrm{Hess}\, f(X, Y) = \nabla^2 f(X, Y) = g(\nabla_X \nabla f, Y)$$
$$\Delta f = Tr(\mathrm{Hess}\, f)$$

Hessian Comparison Theorem

Theorem (Hessian Comparison)

Let $(\mathbb{M}_i, g_i), i \in \{1,2\}$ *be Riemannian manifolds,* $\gamma_i : [0,L] \to \mathbb{M}_i$ *be minimizing geodesics such that*

$$K^{+}_{\mathbb{M}_2}(t) \leq K^{-}_{\mathbb{M}_1}(t)$$

Let $X_i \in \Gamma(T\mathbb{M}_i)$ *be such that for all* $t \in [0,L]$

- $\|X_1(\gamma_1(t))\|_{g_1} = \|X_2(\gamma_2(t))\|_{g_2}$
- $g_1(X_1(\gamma_1(t)), \gamma_1'(t)) = g_2(X_2(\gamma_2(t)), \gamma_2'(t))$

then denoting $p_i = \gamma_i(0), q_i = \gamma_i(t)$,

$$\text{Hess}\, d_{p_1}(X_1(q_1), X_1(q_1)) \leq \text{Hess}\, d_{p_2}(X_2(q_2), X_2(q_2))$$

Rough Sketch of proof:

- Consider variation of geodesics with fixed endpoints; Jacobi fields describe the infinitesimal variation
- Define index $I(X,X) = \int_0^r \langle \nabla_{\dot\gamma} X, \nabla_{\dot\gamma} X\rangle - R(X,\dot\gamma,\dot\gamma,X)\ dt$
- $K_1^+ \leq K_2^- \implies I(X_1,X_1) \leq I(X_2,X_2)$
- Theorem follows from $\nabla^2 d = \alpha I(X,X)$

Rauch Comparison Theorem

Corollary (Rauch Comparison)

Take the same assumptions as in the previous theorem. Then

$$\Delta_1 d_{p_1}(q_2) \leq \Delta_2 d_{p_2}(q_2)$$

This presents a way to compare the behaviors of distance functions, but we still need to something to compare them to.

Model Spaces

Denote by $\mathbb{M}^m_\kappa$ the Riemannian manifold of constant sectional curvature κ and dimension m. Explicitly

$$\mathbb{M}^m_\kappa = \begin{cases} S^m(\kappa) & \kappa > 0 \\ \mathbb{R}^m & \kappa = 0 \\ \mathbb{H}^m(\kappa) & \kappa < 0 \end{cases}$$

We refer to these as Model Spaces. We are able to compute Δd_p explicitly on these spaces, and use this as a basis for comparison.

Laplacian Comparison

Theorem (Laplacian Comparison)

Let $(\mathbb{M}, g)$ *be a Riemannian manifold with dimension* n *and suppose there exists* $\kappa \in \mathbb{R}$ *such that*

$$Ric \geq (n-1)\kappa g$$

Let $p, q \in \mathbb{M}$ *and denote* $r = d(p, q)$*. Then*

$$\Delta d_p(q) \leq \begin{cases} (n-1)\sqrt{\kappa}\cot(\sqrt{\kappa}r) & \kappa > 0 \\ \frac{n-1}{r} & \kappa = 0 \\ (n-1)\sqrt{|\kappa|}\coth(\sqrt{|\kappa|}r) & \kappa < 0 \end{cases}$$

Comparison Function

On the model spaces, the Jacobi fields can be computed explicitly using the Jacobi equation

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}V + R(V,\dot\gamma,\dot\gamma) = 0$$

then the upper bound on $\Delta d_p(q)$ is given by solving an ode.

Basic Definitions

Let $\mathbb{M}$ be a smooth manifold. We say that $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$ is a sub-Riemannian manifold if

- $\mathcal{H}$ is a constant rank, bracket generating subbundle of $T\mathbb{M}$,
- and $g_{\mathcal{H}}$ is a fiberwise inner product on $\mathcal{H}$.

A main goal of sub-Riemannian geometry is to determine adequate notions of curvature that are able to support generalizations of the comparison theorems found in the Riemannian theory.

Some History

- Li-Zelenko 2011, Lee-Li 2013, Agrachev-Lee 2015, Lee-Li-Zelenko 2016: Comparison theorems on Sasakian manifolds
- Rizzi-Silveira 2015, 2017, Barilari-Rizzi 2016: Comparison theorems in 3 Sasakian case
- Baudoin-Bonnefont-Garofalo 2014, Baudoin-Grong-Kuwada-Thalmaier 2017: Eulerian approach to comparison theorems on Sasakian and 3 Sasakian manifolds

sR Manifolds with Metric Complement

Let $\mathbb{M}$ be a smooth manifold. We say that $(\mathbb{M}, \mathcal{H}, g)$ is a sub-Riemannian manifold with metric preserving complement or sRmc-manifold if

- $(\mathbb{M}, g)$ is a Riemannian manifold,
- the metric orthogonally splits as $g = g_{\mathcal{H}} \oplus g_{\mathcal{V}}$,
- and $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$ is a sub-Riemannian manifold.

We denote by $\mathcal{V}$ the orthogonal complement of $\mathcal{H}$ by g.

Motivating Example: Hopf Fibration

Consider $\mathbb{S}^{2n+1}$ foliated as

$$\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1} \xrightarrow{\pi} \mathbb{C}P^n$$

Define the vertical distribution as tangent to the leaves , $\mathbb{S}^1$,

Then setting $\mathcal{H}$ to be orthogonal to $\mathcal{V}$ will make $(\mathbb{S}^{2n+1}, \mathcal{H}, g)$ a sRmc-manifold.

Gromov-Hausdorff Convergence

For a sRmc-manfiold $(\mathbb{M}, \mathcal{H}, g)$ we define the canonical variation of the metric

$$g_\varepsilon = g_{\mathcal{H}} + \frac{1}{\varepsilon} g_{\mathcal{V}}$$

which in the Gromov-Hausdorff sense

$$(\mathbb{M}, \mathcal{H}, g_\varepsilon) \xrightarrow{\varepsilon \to 0^+} (\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$$

The idea is to consider the convergence of Riemannian structures to the sub-Riemannian one.

Hladky-Bott Connection

Theorem (Hladky '12 [5])

There exists a unique metric connection ∇ *on* $(\mathbb{M}, \mathcal{H}, g)$ *such that*

1. $\mathcal{H}$ *and* $\mathcal{V}$ *are* ∇*-parallel,*
2. *The torsion* T *of* ∇ *satisfies*
   - $T(\mathcal{H}, \mathcal{H}) \subset \mathcal{V}$,
   - $T(\mathcal{V}, \mathcal{V}) \subset \mathcal{H}$
3. *For every* $X, Y \in \Gamma(\mathcal{H}), Z, V \in \Gamma(\mathcal{V})$,
   - $\langle T(X, Z), Y\rangle_{\mathcal{H}} = \langle T(Y, Z), X\rangle_{\mathcal{H}}$
   - $\langle T(Z, X), V\rangle_{\mathcal{V}} = \langle T(V, X), Z\rangle_{\mathcal{V}}$.

This is called the Hladky-Bott connection.

Hladky-Bott Connection

We can explicitly write ∇ in terms of the Levi-Civita connection ∇^g as

$$\nabla_X Y = \begin{cases} \pi_{\mathcal{H}} \nabla^g_X Y & X, Y \in \Gamma(\mathcal{H}) \\ \pi_{\mathcal{H}}[X, Y] + A_X Y & Y \in \Gamma(\mathcal{H}), X \in \Gamma(\mathcal{V}) \\ \pi_{\mathcal{V}}[X, Y] + A_X Y & Y \in \Gamma(\mathcal{V}), X \in \Gamma(\mathcal{H}) \\ \pi_{\mathcal{V}} \nabla^g_X Y & X, Y \in \Gamma(\mathcal{V}) \end{cases}$$

where the tensor A is defined by

$$\langle A_X Y, Z \rangle = \frac{1}{2} \left((\mathcal{L}_{X_{\mathcal{V}}} g)(Y_{\mathcal{H}}, Z_{\mathcal{H}}) + (\mathcal{L}_{X_{\mathcal{H}}} g)(Y_{\mathcal{V}}, Z_{\mathcal{V}}) \right)$$

Bundle-like Metrics and Totally Geodesic Foliations

There are two important properties we will require:

- Bundle-like metric: A foliation is said to have a bundle-like metric if the metric locally splits orthogonally. This is equivalent to

$$\mathcal{L}_{\mathcal{V}}g(\mathcal{H},\mathcal{H}) = 0$$

- Totally geodesic foliation: A foliation is said to be totally geodesic if the geodesics of the fibers are embedded geodesics of the total space. This is equivalent to

$$\mathcal{L}_{\mathcal{H}}g(\mathcal{V},\mathcal{V}) = 0$$

J Map

On $(\mathbb{M}, \mathcal{H}, g)$ we can associate to each vector field $Z \in \Gamma(T\mathbb{M})$ an endomorphism J_Z of $T\mathbb{M}$ defined by

$$\langle J_Z X, Y \rangle = \langle Z, T(X, Y) \rangle$$

If $\mathcal{V}$ is integrable,

$$\begin{cases} J_Z X \in \mathcal{H} & \text{if } Z \in \mathcal{V}, X \in \mathcal{H} \\ J_Z X = 0 & \text{otherwise} \end{cases}$$

We thus take the perspective

$$J \colon \mathcal{V} \to \mathrm{End}(\mathcal{H}), \quad Z \mapsto J_Z$$

H-type Foliations

Definition

Let $(\mathbb{M}, \mathcal{H}, g)$ be a sRmc-manfiold. We say that $(\mathbb{M}, \mathcal{H}, g, J)$ is an H-type foliation if

1. $(\mathbb{M}, \mathcal{V}, g)$ is a totally geodesic foliation with bundle-like metric, and
2. for all $X, Y \in \Gamma(\mathcal{H}), Z \in \Gamma(\mathcal{V})$,

$$\langle J_Z X, J_Z Y \rangle_{\mathcal{H}} = \|Z\|^2 \langle X, Y \rangle_{\mathcal{H}}$$

Parallel Torsion

We also refine the definition of H-type foliations based on the behavior of derivatives of the Hladky-Bott torsion T.

- If $\delta_{\mathcal{H}} T = 0$ we say $\mathbb{M}$ is of Yang-Mills type,
- If $\nabla_{\mathcal{H}} T = 0$ we say $\mathbb{M}$ has horizontally parallel torsion, and
- If $\nabla T = 0$ we say $\mathbb{M}$ has completely parallel torsion.

Lemma

All H-type foliations are Yang-Mills.

J^2 condition

In many model cases, such as the Complex-, Quaternionic-, and Octonionic-Hopf fibrations we have a stronger property:

Definition

Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation. We say that it satisfies the J^2 condition if for every $Z_1, Z_2 \in \mathcal{V}$ with $\langle Z_1, Z_2 \rangle = 0$ there exists $Z_3 \in \mathcal{V}$ such that

$$J_{Z_1} J_{Z_2} = J_{Z_3}$$

The H-type groups with this property were classified by (M. Cowling, A.H. Dooley, A. Korányi, and F.Ricci '91 [4])

Metric Connections, Jacobi Equation

For the remainder of the talk, let $(M, \mathcal{H}, g_\varepsilon)$

- be a sRmc-manifold,
- equipped with the canonical variation g_ε,
- having horizontally parallel torsion,
- and satisfying the J^2 condition.

Jacobi Equation

There's a subtlety that needs to be considered. The Levi-Civita connection ∇ is self-adjoint.

As a consequence the Jacobi equation is simply

$$\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}W + R(W, \dot{\gamma})\dot{\gamma} = 0$$

but this isn't true for general connections, or in particular the Bott connection.

Adjoint Connections and the Jacobi Equation

For an arbitrary connection ∇ with torsion

$$Tor(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

define its adjoint connection to be

$$\hat{\nabla}_X Y = \nabla_X Y - Tor(X, Y)$$

notice, $\hat{\hat{\nabla}} = \nabla$.

In general, the adjoint of a metric connection is not metric. As a consequence, terms involving the torsion of ∇ are introduced to the Jacobi equation.

Jacobi Equation for Metric Adjoint Connections

However, in the special case that both $\nabla, \hat{\nabla}$ are metric, the Jacobi equation along a geodesic γ is

$$\hat{\nabla}_{\dot{\gamma}} \nabla_{\dot{\gamma}} W + \hat{R}(W, \dot{\gamma})\dot{\gamma} = 0$$

This is a consequence of the commutation $\nabla_V \dot{\gamma} = \hat{\nabla}_{\dot{\gamma}} V$ for a Jacobi field V along a geodesic γ.

ε-invariant Metric Adjoint Connection

For any $\varepsilon > 0$ the connection

$$\hat{\nabla}^{\varepsilon}_X Y = \nabla_X Y + J^{\varepsilon}_X Y$$

will be metric with metric adjoint ∇^{ε}.

From this we recover a Jacobi equation for all $\epsilon > 0$.

The Comparison Principle

Theorem (Baudoin, Grong, Kuwada, & Thalmaier '17 [1])

- *Let* $x, y \in \mathbb{M}$,
- $\gamma \colon [0, r_\varepsilon] \to \mathbb{M}$ *a unit speed* g_ε*-geodesic connecting* x, y, *and*
- $W_1, \cdots, W_k$ *be a collection of vector fields along* γ *such that*

$$\sum_{i=0}^{k} \int_0^{r_\varepsilon} \langle \hat{\nabla}^\varepsilon_{\dot\gamma} \nabla^\varepsilon_{\dot\gamma} W_i + \hat{R}^\varepsilon(W_i, \dot\gamma)\dot\gamma, W_i \rangle_\varepsilon \geq 0$$

then at $y = \gamma(r_\varepsilon)$ *it holds that*

$$\sum_{i=0}^{k} \mathrm{Hess}^{\hat{\nabla}^\varepsilon}(d_p^\varepsilon)(W_i, W_i) \leq \sum_{i=0}^{k} \langle W_i, \hat{\nabla}^\varepsilon_{\dot\gamma} W_i \rangle_\varepsilon$$

with equality if and only if the W_i *are Jacobi fields.*

- Along a geodesic γ let V satisfy the Jacobi equation

$$\hat{\nabla}_{\dot{\gamma}} \nabla_{\dot{\gamma}} V - \hat{R}(V, \dot{\gamma})\dot{\gamma} = 0$$

and initial conditions $V(0) = 0, V(r) = X$

- Then it can be shown

$$\hat{\nabla}^2 d_p(q)(X, X) = I(V, V)$$

with (ε-invariant) index

$$I(V, V) = \int_0^r \langle \hat{\nabla}^{\varepsilon}_{\dot{\gamma}} V, \nabla^{\varepsilon}_{\dot{\gamma}} V \rangle - \hat{R}^{\varepsilon}(V, \dot{\gamma}, \dot{\gamma}, V)\, dt$$

- This gives bounds on the behavior of $\mathrm{Hess}^{\hat{\nabla}^{\varepsilon}}(r_{\varepsilon})$

Horizontal Splitting

We introduce an orthogonal splitting of the horizontal bundle. Fixing a vector field $Y \in \mathcal{H}$,

$$\mathcal{H} = \text{span}(Y) \oplus \mathcal{H}_{Riem}(Y) \oplus \mathcal{H}_{Sas}(Y)$$

where

$$\mathcal{H}_{Sas}(Y) = \{J_Z Y | Z \in \mathcal{V}\}$$

$$\mathcal{H}_{Riem}(Y) = \{X \in \mathcal{H} | X \perp \mathcal{H}_{Sas} \oplus \text{span}(Y)\}$$

Lemma

Denoting $n = \text{rk}(\mathcal{H}), m = \text{rk}(\mathcal{V})$, *we will have*

$$\dim(\mathcal{H}_{Sas}) = m, \quad \dim(\mathcal{H}_{Riem}) = n - m - 1$$

Comparison Functions

Simliarly to the Riemannian case, we consider the comparision functions

$$F_{Riem}(r,\kappa)=\begin{cases}\sqrt{\kappa}\cot(\sqrt{\kappa}r) & \text{if } \kappa>0\\ \frac{1}{r} & \text{if } \kappa=0\\ \sqrt{|\kappa|}\coth(\sqrt{|\kappa|}r) & \text{if } \kappa<0\end{cases}$$

$$F_{Sas}(r,\kappa)=\begin{cases}\frac{\sqrt{\kappa}\left(\sin(\sqrt{\kappa}r)-\sqrt{\kappa}r\cos(\sqrt{\kappa}r)\right)}{2-2\cos(\sqrt{\kappa}r)-\sqrt{\kappa}r\sin(\sqrt{\kappa}r)} & \text{if } \kappa>0\\ \frac{4}{r} & \text{if } \kappa=0\\ \frac{\sqrt{\kappa}\left(\sqrt{\kappa}r\cosh(\sqrt{\kappa}r)-\sinh(\sqrt{\kappa}r)\right)}{2-2\cosh(\sqrt{\kappa}r)+\sqrt{\kappa}r\sinh(\sqrt{\kappa}r)} & \text{if } \kappa<0\end{cases}$$

These comparison functions will correspond to the splitting of $\mathcal{H}$.

Hessian Comparisons

Theorem (Baudoin, Grong, M., & Rizzi '19 [3])

- *Let* $\gamma : [0, r_\varepsilon] \to \mathbb{M}$ *be a* g_ε*-geodesic. Then*

$$\mathrm{Hess}(r_\varepsilon)(\dot\gamma, \dot\gamma) \leq \frac{\|\dot\gamma\|^2 \left(1 - \|\dot\gamma\|^2\right)}{r_\varepsilon}$$

- *If* $\mathrm{Sec}(X \wedge Y) \geq \rho > 0$ *for all unit* $X, Y \in \mathcal{H}_{Riem}(\dot\gamma)$*, then*

$$\mathrm{Hess}(r_\varepsilon)(X, X) \leq F_{Riem}(r_\varepsilon, K)$$

- *If* $\mathrm{Sec}(X \wedge J_Z X) \geq \rho > 0$ *for all unit* $X \in \mathcal{H}_{Sas}(\dot\gamma)$*, then*

$$\mathrm{Hess}(r_\varepsilon)(X, X) \leq F_{Sas}(r_\varepsilon, K)$$

Where K *is a constant depending on* $\rho, \varepsilon, \|\nabla_{\mathcal{V}} r_\varepsilon\|$*, and* $\|\nabla_{\mathcal{H}} r_\varepsilon\|$*.*

Horizontal Ricci Curvature

We define the horizontal Ricci curvature as the horizontal trace of the Riemann tensor,

$$\begin{aligned}\mathrm{Ric}_{\mathcal{H}}(X,X) &= \sum_{i=0}^{n} \langle R^{\nabla}(W_i,X)X, W_i\rangle \\ &= \langle R^{\nabla}(Y,X)X, Y\rangle + \mathrm{Ric}_{Sas}(X,X) + \mathrm{Ric}_{Riem}(X,X)\end{aligned}$$

where the splitting corresponds to the decomposition

$$\mathcal{H} = \mathrm{span}(Y) \oplus \mathcal{H}_{Sas} \oplus \mathcal{H}_{Riem}$$

Bonnet-Meyers Estimates

Theorem (Baudoin, Grong, M., & Rizzi '19 [3])

Let $\rho > 0$. Then for unit $X \in \mathcal{H}$,

1. $$\frac{\mathrm{Ric}_{Riem}(X,X)}{n-m-1} \geq \rho \implies \mathrm{diam}_0(\mathbb{M}) \leq \frac{\pi}{\sqrt{\rho}}$$
2. $$\mathrm{Sec}(X \wedge J_Z X) \geq \rho \implies \mathrm{diam}_0(\mathbb{M}) \leq \frac{2\pi}{\sqrt{\rho}}$$
3. $$\frac{\mathrm{Ric}_{Sas}(X,X)}{m} \geq \rho \implies \mathrm{diam}_0(\mathbb{M}) \leq \frac{2\pi\sqrt{3}}{\sqrt{\rho}}$$

and in each case the fundamental group of $\mathbb{M}$ must be finite.

The first two of these are sharp, as they are achieved in the complex, quaternionic, and octonionic Hopf fibrations.

sub-Laplacian

Similarly to the horizontal Ricci curvature, we can define the sub-Laplacian as the trace of the Hessian. For the distance function r_ε along a geodesic γ with $Y = \nabla_{\mathcal{H}} r_\varepsilon$,

$$\Delta_{\mathcal{H}} r_\varepsilon = \sum_{i=0}^{n} \text{Hess}(r_\varepsilon)(W_i, W_i)$$

$$= \text{Hess}(r_\varepsilon)(Y, Y) + \sum_{i=0}^{m} \text{Hess}(r_\varepsilon)(J_{Z_i} Y, J_{Z_i} Y) + \sum_{i=0}^{n-m-1} \text{Hess}(r_\varepsilon)(W_i, W_i)$$

for appropriate bases $\{W_i\}$ of $\mathcal{H}$ and $\{Z_i\}$ of $\mathcal{V}$. This splitting corresponds again to the decomposition

$$\mathcal{H} = \text{span}(Y) \oplus \mathcal{H}_{Sas} \oplus \mathcal{H}_{Riem}$$

Laplacian Comparisons

In each component of the horizontal decomposition we can use the previous comparisons on the Hessian to obtain

Theorem (Baudoin, Grong, M., & Rizzi '19 [3])

Let $(\mathbb{M}, g, \mathcal{H})$ be an H-type foliation with parallel horizontal Clifford structure and satisfying the J^2 condition, and with nonnegative horizontal Bott curvature. Then there exists a $C > 4$ such that

$$\Delta_{\mathcal{H}} r_0 \leq \frac{n - m + 3 + C(m-1)}{r_0}$$

This is not sharp, but we can recover sharp estimates in each subspace.

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Thank you for your attention!