

# On Holonomy: An invitation to Riemannian Geometry

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- Riemannian metric

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- 3 Locally Euclidean

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## Definition

If there is a collection  $\Phi$  of smooth (infinitely differentiable) transition maps that cover  $\mathbb{M}$ , we call  $(\mathbb{M}, \mathcal{T}, \Phi)$  a smooth manifold.

# The Tangent Bundle

- Associated to every point  $p \in \mathbb{M}$  we have a copy of  $\mathbb{R}^n$ . We call these tangent spaces  $T_p\mathbb{M}$ , and their disjoint union the tangent bundle  $T\mathbb{M} = \cup_{p \in \mathbb{M}} T_p\mathbb{M}$ .

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- The vectors in  $T_p\mathbb{M}$  can be put into 1-1 correspondence with the smooth curves  $\gamma: [0, 1] \rightarrow \mathbb{M}$  passing through  $p$  “in the same direction”, so we view elements of  $T_p\mathbb{M}$  as tangent vectors to curves.



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- If at every point  $p$  in  $\mathbb{M}$  we choose a tangent vector  $X_p \in T_p\mathbb{M}$ , we call the collection a vector field  $X \in \mathfrak{X}(\mathbb{M})$ .

# Riemannian Manifolds

- In order to make sense of length on the tangent spaces, we let

$$g_p(\cdot, \cdot): T_p\mathbb{M} \times T_p\mathbb{M} \rightarrow \mathbb{R}$$

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- If these inner products vary smoothly as a function of  $p$ , we call it a Riemannian metric denoted  $g$ .
- A smooth manifold with a Riemannian metric  $(\mathbb{M}, \mathcal{T}, \Phi, g)$  is called a Riemannian manifold.

# Tangent Plane Identification Problem

Suppose we have two points  $p, q \in \mathbb{M}$ , and a vector  $X_p \in T_p\mathbb{M}$ .  
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We can visualize connecting  $p$  and  $q$  by a curve, and “transporting”  $X_p$  along the curve to  $q$ , but it is not clear how to do this.

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Any such  $\nabla$  is called a connection.

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Given a curve  $\gamma$  from  $p$  to  $q$  and a  $\gamma$ -parallel vector field  $X$ , we say that  $X_q$  is the parallel transport of  $X_p$  along  $\gamma$ , which we denote

$$\begin{aligned} \tau_\gamma: T_p\mathbb{M} &\rightarrow T_q\mathbb{M} \\ X_p &\mapsto \tau_\gamma X_p = X_q \end{aligned}$$

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## Theorem (Fundamental Theorem of Riemannian Geometry)

*There exists a unique connection  $\nabla$  such that*

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This connection is called the Levi-Civita connection.

# Riemann Curvature Tensor

It will be important later to discuss the idea of curvature, which is quantified by the Riemann curvature tensor

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$$

This is a tensor (not obvious) and measures the noncommutativity of the connection  $\nabla$ .

# Tangent Space Identification?

However, this process still has not answered the problem of identifying the tangent vectors: for each curve from  $p$  to  $q$  we have an identification of  $T_p\mathbb{M}$  with  $T_q\mathbb{M}$ .

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While this has no simple resolution, it introduces an interesting question. **What are all possible identifications of the tangent spaces?**

# Holonomy

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## Definition

Fix a point  $p \in \mathbb{M}$  and consider all possible curves  $\gamma$  in  $\mathbb{M}$  that both start and end at  $p$ . Associated to each one there is an automorphism of  $T_p\mathbb{M}$ . We call the collection of automorphisms the Riemannian holonomy group of  $\mathbb{M}$  at  $p$ , denoted  $\text{Hol}(\mathbb{M}, p)$ .



# Immediate Properties of the Holonomy Group

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and so the holonomy group  $\text{Hol}(\mathbb{M})$  is independent of the basepoint.

- Since parallel transport is a linear isometry, it is clear that

$$\text{Hol}(\mathbb{M}) \subseteq O(n)$$

# Restricted Holonomy and the Fundamental Group

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- 3  $\nabla$  is flat if and only if  $\text{Hol}^0(\mathbb{M})$  is trivial.



Example:  $\mathbb{R}^2 \times S^1$ 

Consider  $M = \mathbb{R}^2 \times S^1$ , which can be constructed as

$$\mathbb{R}^2 \times S^1 \cong \mathbb{R}^2 \times [0, 1] / \sim$$

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- However,  $\text{Hol}(\mathbb{M}) \subseteq O(2)$  is the group generated by  $f_\alpha$ ; finite if  $\alpha = k\pi$ ,  $k \in \mathbb{Q}$ , and isomorphic to  $\mathbb{Z}$  otherwise.

# Orientability and $SO(n)$

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- It follows that if  $(\mathbb{M}, g)$  is orientable, it must hold that

$$\text{Hol}(\mathbb{M}) \subseteq SO(n)$$

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- By the Newlander-Nirenberg Theorem,  $(\mathbb{M}, g)$  is Kähler.

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- 2 *There exists on  $(\mathbb{M}, g)$  a tensor field  $\alpha$  of type  $(r,s)$  which has vanishing covariant derivative:  $\nabla\alpha = 0$ .*

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- 3 There exists  $p \in \mathbb{M}$  and on  $T_p^{(r,s)}\mathbb{M}$  a tensor  $\alpha$  invariant by  $\text{Hol}(\mathbb{M})$ .

# Calabi-Yau manifolds and $SU(n)$

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- Such manifolds are called Calabi-Yau, and are necessarily Ricci-flat.

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- Again applying the fundamental principle,

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- Equivalently, at every point  $(p_1, p_2) \in M_1 \times M_2$ , the tangent plane splits as the Euclidean product:

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- A Riemannian manifold (locally) isometric to a Riemannian product is called (locally) reducible.



# deRham Decomposition

## Theorem (deRham)

*Let  $(\mathbb{M}, g)$  be a complete, simply connected Riemannian manifold. Then  $\text{Hol}(\mathbb{M})$  is reducible as a direct product if and only if  $(\mathbb{M}, g)$  is reducible as a Riemannian product.*

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That is, the splitting

$$(\mathbb{M}, g) = (\mathbb{M}_1 \times \mathbb{M}_2, g_1 \times g_2)$$

is equivalent to the splitting

$$\text{Hol}(\mathbb{M}, g) = \text{Hol}(\mathbb{M}_1, g_1) \times \text{Hol}(\mathbb{M}_2, g_2)$$

# Lassos and Ambrose-Singer

By considering “lassos”, that is small loops conjugated by parallel transport, we have the following:

## Theorem (Ambrose-Singer)

*The Lie algebra  $\mathfrak{hol}(p)$  of  $\text{Hol}(\mathbb{M})$  at  $p \in \mathbb{M}$  is the subalgebra of  $\mathfrak{so}(T_p\mathbb{M})$  generated by the elements*

$$\tau_\gamma^{-1} \circ R(\tau_\gamma X_p, \tau_\gamma Y_p) \circ \tau_\gamma$$

*where  $X_p, Y_p$  run through  $T_p\mathbb{M}$  and  $\gamma$  runs through all paths starting at  $p$ .*

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- Using these two ideas and Cartan's classification of irreducible linear real representations of real Lie groups, Berger classified all possible Riemannian manifolds by their holonomy:

## Theorem (Berger)

*Suppose  $(M, g)$  is a Riemannian manifold such that  $\text{Hol}^0(M)$  is irreducible. Then at least one of the following is satisfied:*

- $\text{Hol}^0(M)$  acts transitively on the sphere, or
- $M$  is a locally symmetric space of rank greater than or equal to 2.

# Symmetric Case

## Definition

A Riemannian manifold  $\mathbb{M}$  is called symmetric if for each point  $p \in \mathbb{M}$  there exists an isometry  $f_p: \mathbb{M} \rightarrow \mathbb{M}$  such that

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There is a complete classification due to Cartan of all symmetric Riemannian spaces.

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This relates to the holonomy group in the following way:

## Theorem

*For an irreducible simply connected symmetric manifold  $\mathbb{M} = G/H$ , we have that  $\text{Hol}(\mathbb{M}) = H$  acting by the adjoint representation.*

# Nonsymmetric Case

## Theorem (Berger, Simons)

For an irreducible simply connected nonsymmetric manifold  $\mathbb{M}$ , one of the following cases occurs:

$\dim(\mathbb{M})$	$\text{Hol}^0(\mathbb{M})$	Type
$n$	$O(n)$	Generic
$n$	$SO(n)$	Oriented
$n = 2m$	$U(m)$	Kähler
$n = 2m$	$SU(m)$	Calabi-Yau
$n = 4m$	$Sp(m) \cdot Sp(1)$	Quaternion-Kähler
$n = 4m$	$Sp(m)$	Hyperkähler
$n = 7$	$G_2$	
$n = 8$	$Spin(7)$	

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- 1 For a curve  $c(t)$  with  $c(0) = p$ ,  $c'(0) = v$ ,

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- 4 Since  $Z = 0$ ,  $(\nabla R)_p = 0$ . This can be easily extended to  $\nabla R = 0$  everywhere, and so  $\mathbb{M}$  is symmetric.

## Further Questions

- It is possible to instead consider the holonomy associated to an arbitrary connection (not Levi-Civita) on a manifold, or even the holonomy of abstract vector bundles.

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- The holonomy of a manifold is closely related to its curvature tensor  $R$ , and so it is possible to consider the possible curvatures given a holonomy group, or vice-versa.
- It's an open question whether  $\text{Hol}(\mathbb{M})$  can be noncompact for compact manifolds  $\mathbb{M}$ .