# Holonomy of H-type Foliations

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## Basic Definitions, sub-Riemannian Geometry

Let $\mathbb{M}$ be a smooth manifold. We say that

- $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$ is a sub-Riemannian manifold if
  - $\mathcal{H}$ is a constant rank, bracket generating subbundle of $T\mathbb{M}$,
  - and $g_{\mathcal{H}}$ is a inner product on $\mathcal{H}$.
- $(\mathbb{M}, \mathcal{H}, g)$ is a sub-Riemannian manifold with metric preserving complement or sRmc-manifold if
  - $(\mathbb{M}, g)$ is a Riemannian manifold,
  - the metric orthogonally splits as $g = g_{\mathcal{H}} \oplus g_{\mathcal{V}}$,
  - and $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$ is a sub-Riemannian manifold.

  We denote by $\mathcal{V}$ the orthogonal complement of $\mathcal{H}$ by $g$.

# sub-Riemannian Geometry versus Riemannian Geometry

- Riemannian geometry is distinguished by the Riemannian metric $g$ and thus a definition of distance in all directions.
- sub-Riemannian geometry has a notion of distance, but only in some directions.
- A major goal of sub-Riemannian geometry is to recover Riemannian results (or generalizations).
- The Chow-Rashevskii theorem guarantees that if the horizontal bundle is bracket generating, then any two points are connected by a horizontal path.

# Motivating Examples

There is still hope to understand sub-Riemannian geometry through comparison with models.

- The Heisenberg group is $\mathbb{R}^3$ equipped with vector fields

$$X = \partial_x - \frac{1}{2} y \partial_z, \qquad Y = \partial_y + \frac{1}{2} x \partial_z, \tag{1}$$

  setting $\mathcal{H} = \mathrm{Span}\{X, Y\}$ and defining $g_{\mathcal{H}}$ so that $X, Y$ are orthonormal.

- We can equivalently see this as induced by a submersion

$$\mathbb{R} \hookrightarrow \mathbb{R}^3 \to \mathbb{C} \tag{2}$$

# Motivating Examples

- The Hopf fibration is the sphere $S^3$ equipped with horizontal distribution induced by the submersion

$$S^1 \hookrightarrow S^3 \to \mathbb{C}P^1 \tag{3}$$

- The Anti-de Sitter space is the hyperbolic space $\mathrm{H}^3$ equipped with horizontal distribution induced by the submersion

$$S^1 \hookrightarrow \mathrm{H}^3 \to \mathrm{H}P^1 \tag{4}$$

# Gromov-Hausdorff Convergence

For a sRmc-manfiold $(\mathbb{M}, \mathcal{H}, g)$ we define the canonical variation of the metric

$$g_\varepsilon = g_{\mathcal{H}} + \frac{1}{\varepsilon} g_{\mathcal{V}}$$

which in the Gromov-Hausdorff sense

$$(\mathbb{M}, \mathcal{H}, g_\varepsilon) \xrightarrow{\varepsilon \to 0^+} (\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$$

The key idea of this project is to leverage our knowledge of the Riemannian structure to understand the sub-Riemannian one.

# Hladky-Bott Connection

**Theorem (Hladky '12)**

*There exists a unique metric connection $\nabla$ on $(\mathbb{M}, \mathcal{H}, g)$ such that*

1. $\mathcal{H}$ *and* $\mathcal{V}$ *are* $\nabla$*-parallel,*
2. *The torsion* $T$ *of* $\nabla$ *satisfies*
   - $T(\mathcal{H}, \mathcal{H}) \subset \mathcal{V}$,
   - $T(\mathcal{V}, \mathcal{V}) \subset \mathcal{H}$
3. *For every* $X, Y \in \Gamma(\mathcal{H}), Z, V \in \Gamma(\mathcal{V})$,
   - $\langle T(X,Z), Y\rangle_{\mathcal{H}} = \langle T(Y,Z), X\rangle_{\mathcal{H}}$
   - $\langle T(Z,X), V\rangle_{\mathcal{V}} = \langle T(V,X), Z\rangle_{\mathcal{V}}$.

This is called the Hladky-Bott connection.

# Hladky-Bott Connection

We can explicitly write $\nabla$ in terms of the Levi-Civita connection $\nabla^g$ as

$$\nabla_X Y = \begin{cases} \pi_{\mathcal{H}} \nabla^g_X Y & X, Y \in \Gamma(\mathcal{H}) \\ \pi_{\mathcal{H}}[X, Y] + A_X Y & Y \in \Gamma(\mathcal{H}), X \in \Gamma(\mathcal{V}) \\ \pi_{\mathcal{V}}[X, Y] + A_X Y & Y \in \Gamma(\mathcal{V}), X \in \Gamma(\mathcal{H}) \\ \pi_{\mathcal{V}} \nabla^g_X Y & X, Y \in \Gamma(\mathcal{V}) \end{cases}$$

where the tensor $A$ is defined by

$$\langle A_X Y, Z \rangle = \frac{1}{2} \left( (\mathcal{L}_{X_{\mathcal{V}}} g)(Y_{\mathcal{H}}, Z_{\mathcal{H}}) + (\mathcal{L}_{X_{\mathcal{H}}} g)(Y_{\mathcal{V}}, Z_{\mathcal{V}}) \right)$$

# Bundle-like Metrics and Totally Geodesic Foliations

There are two important properties we will require:

- Bundle-like metric: A foliation is said to have a bundle-like metric if the metric locally splits orthogonally. This is equivalent to

$$\mathcal{L}_{\mathcal{V}} g(\mathcal{H}, \mathcal{H}) = 0$$

- Totally geodesic foliation: A foliation is said to be totally geodesic if the geodesics of the fibers are embedded geodesics of the total space. This is equivalent to

$$\mathcal{L}_{\mathcal{H}} g(\mathcal{V}, \mathcal{V}) = 0$$

# $J$ Map

On $(\mathbb{M}, \mathcal{H}, g)$ we can associate to each vector field $Z \in \Gamma(T\mathbb{M})$ an endomorphism $J_Z$ of $T\mathbb{M}$ defined by

$$\langle J_Z X, Y \rangle = \langle Z, T(X, Y) \rangle$$

If $\mathcal{V}$ is integrable,

$$\begin{cases} J_Z X \in \mathcal{H} & \text{if } Z \in \mathcal{V}, X \in \mathcal{H} \\ J_Z X = 0 & \text{otherwise} \end{cases}$$

We thus take the perspective

$$J \colon \mathcal{V} \to \mathrm{End}(\mathcal{H}), \quad Z \mapsto J_Z$$

# H-type Foliations

**Definition**

Let $(\mathbb{M}, \mathcal{H}, g)$ be a sRmc-manfiold. We say that $(\mathbb{M}, \mathcal{H}, g, J)$ is an H-type foliation if

1. $(\mathbb{M}, \mathcal{V}, g)$ is a totally geodesic foliation with bundle-like metric,
2. $\mathcal{V}$ is integrable, and
3. for all $X, Y \in \Gamma(\mathcal{H}), Z \in \Gamma(\mathcal{V})$,

$$\langle J_Z X, J_Z Y \rangle_{\mathcal{H}} = \|Z\|^2 \langle X, Y \rangle_{\mathcal{H}}$$

# Parallel Torsion

We also refine the definition of H-type foliations based on the behavior of derivatives of the Hladky-Bott torsion $T$.

- If $\delta_{\mathcal{H}} T = 0$ we say $\mathbb{M}$ is of Yang-Mills type,
- If $\nabla_{\mathcal{H}} T = 0$ we say $\mathbb{M}$ has horizontally parallel torsion, and
- If $\nabla T = 0$ we say $\mathbb{M}$ has completely parallel torsion.

**Lemma**

*All H-type foliations are Yang-Mills.*

# Examples

| **Structure** | Torsion |
|---|---|
| **Complex Type,** $m = 1, n = 2k$ | |
| K-Contact Manifolds | YM |
| Heisenberg Group, Hopf, Anti de-Sitter Fibrations | CP |
| **Twistor Type,** $m = 2, n = 4k$ | |
| Twistor space over quaternionic Kähler manifold | HP |
| Projective Twistor space $\mathbb{CP}^1 \hookrightarrow \mathbb{C}P^{2k+1} \to \mathbb{H}P^k$ | HP |
| Hyperbolic Twistor space $\mathbb{CP}^1 \hookrightarrow \mathbb{C}H^{2k+1} \to \mathbb{H}H^k$ | HP |
| **Quaternionic Type,** $m = 3, n = 4k$ | |
| 3-Sasakian Manifolds | HP |
| Torus bundle over hyperkähler manifolds | CP |
| Quaternionic Heisenberg Group, Hopf, and Anti-de Sitter Fibrations | CP/HP |
| **Octonionic Type,** $m = 7, n = 8$ | |
| Octonionic Heisenberg Group, Hopf, and Anti-de Sitter Fibrations | CP/HP |
| **H-type Groups**, $m$ is arbitrary | CP |

# Clifford Structures

Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation with $Z_i, Z_j \in \mathcal{V}$, then

$$J_{Z_i} J_{Z_j} + J_{Z_j} J_{Z_i} = -2\langle Z_i, Z_j \rangle Id_{\mathcal{H}}$$

and so we can extend $J$ in the natural way to

$$J \colon Cl(\mathcal{V}) \to \mathrm{End}(\mathcal{H})$$

There is a classification of such Clifford structures over Riemannian manifolds. (A. Moroianu, U. Semmelmann '11)

# Parallel Horizontal Clifford Structures

Arising from Riemannian or semi-Riemannian foliations with curvature constancy we have the following notion:

**Definition**

Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation with horizontally parallel torsion. Then if there exists a map

$$\Psi \colon \mathcal{V} \times \mathcal{V} \to Cl_2(\mathcal{V})$$

such that

$$(\nabla_{Z_1} J)_{Z_2} = J_{\Psi(Z_1, Z_2)}$$

for all $Z_1, Z_2 \in \Gamma(\mathcal{V})$ then we say that $\mathbb{M}$ has a parallel horizontal Clifford structure.

This has a strong relationship with the horizontal holonomy of the space.

In the case that the H-type foliation $(\mathbb{M}, \mathcal{H}, g)$ is induced by a global submersion

$$F \hookrightarrow \mathbb{M} \to \mathbb{B}$$

we refer to the structure $(\mathbb{M}, \mathcal{H}, g, \mathbb{B})$ as an H-type submersion with fiber $F$.

- These are models for the H-type foliations in the sense that every foliation is locally a submersion.
- We have a complete classification of H-type submersions with horizontally parallel Clifford structure.

# Classification of H-type submersions with horizontally parallel Clifford structure, $\kappa > 0$

| $\mathbb{M}$ | $\mathbb{B}$ | Fiber | rank($\mathcal{H}$) | rank($\mathcal{V}$) |
|---|---|---|---|---|
| Twistor space | Quaternion-Kähler with positive scalar curvature | $\mathbb{S}^2$ | $4k$ | 2 |
| 3-Sasakian | Quaternion-Kähler with positive scalar curvature | $\mathbb{S}^3$ | $4k$ | 3 |
| Quaternion-Sasakian | Product of two quaternion-Kähler with positive scalar curvature | $\mathbb{R}P^3$ | $4k$ | 3 |
| $\frac{Sp(q^++1)\times Sp(q^-+1)}{Sp(q^+)\times Sp(q^-)\times Sp(1)}$ | $\mathbb{H}P^{q^+}\times\mathbb{H}P^{q^-}$ | $\mathbb{S}^3$ | $4(q^++q^-)$ | 3 |
| $\frac{Sp(k+2)}{Sp(k)\times Spin(4)}$ | $\frac{Sp(k+2)}{Sp(k)\times Sp(2)}$ | $\mathbb{S}^4$ | $8k$ | 4 |
| $\frac{SU(k+4)}{S(U(k)\times Sp(2)U(1))}$ | $\frac{SU(k+4)}{S(U(k)\times U(4))}$ | $\mathbb{R}P^5$ | $8k$ | 5 |
| $\frac{SO(k+8)}{SO(k)\times Spin(7)}$ | $\frac{SO(k+8)}{SO(k)\times SO(8)}$ | $\mathbb{R}P^7$ | $8k$, $k\geq 3$, $k$ odd | 7 |
| $\frac{Spin(k+8)}{SO(k)\times Spin(7)}$ | $\frac{SO(k+8)}{SO(k)\times SO(8)}$ | $\mathbb{S}^7$ | $8k$, $k=1$, $k$ even | 7 |
| Exceptional cases | | | | |
| $\frac{F_4}{Spin(8)}$ | $\frac{F_4}{Spin(9)}=\mathbb{O}P^2$ | $\mathbb{S}^8$ | 16 | 8 |
| $\frac{E_6}{Spin(8)U(1)}$ | $\frac{E_6}{Spin(10)U(1)}=(\mathbb{C}\otimes\mathbb{O})P^2$ | $\mathbb{S}^9$ | 32 | 9 |
| $\frac{E_7}{Spin(11)SU(2)}$ | $\frac{E_7}{Spin(12)SU(2)}=(\mathbb{H}\otimes\mathbb{O})P^2$ | $\mathbb{S}^{11}$ | 64 | 11 |
| $\frac{E_8}{Spin(15)}$ | $\frac{E_8}{Spin^+(16)}=(\mathbb{O}\otimes\mathbb{O})P^2$ | $\mathbb{S}^{15}$ | 128 | 15 |

# Horizontal Holonomy

Because the Bott connection preserves the horizontal distribution (independently of $\varepsilon$) we can understand that H-type foliations have a notion of horizontal holonomy.

**Definition**

Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation. Fix a point $p \in \mathbb{M}$, and consider all loops $\gamma$ starting and ending at $p$. Each curve defines an isometry

$$\tau_\gamma : \mathcal{H}_p \to \mathcal{H}_p$$

the collection of which forms a group under concatenation, called the horizontal holonomy group $\mathrm{Hol}^\nabla(\mathbb{M}, \mathcal{H})$ of $(\mathbb{M}, \mathcal{H}, g)$.

This is independent of the choice of $p \in \mathbb{M}$.

# Horizontal Holonomy of H-type Submersions

In particular, the holonomy of H-type submersions can be understood very concretely:

**Theorem**

*Let* $(\mathbb{M}, \mathcal{H}, g, \mathbb{B})$ *be an H-type submersion. Then*

$$\mathrm{Hol}^{\nabla}(\mathbb{M}, \mathcal{H}) \cong \mathrm{Hol}(\mathbb{B})$$

*where* $\mathrm{Hol}(\mathbb{B})$ *denotes the Riemannian holonomy group of* $\mathbb{B}$ *equipped with the Levi-Civita connection induced by* $g_{\mathcal{H}}$*.*

# List of Horizontal Holonomy Groups

From this, we are able to ascertain a complete list of horizontal holonomy groups that occur for H-type submersions; For those with $\kappa > 0$:

$$\begin{gathered} \mathrm{Sp}(n)\mathrm{Sp}(1) \\ \mathrm{Sp}(n) \times \mathrm{Sp}(2) \\ S(U(n) \times U(4)) \\ SO(n) \times SO(8) \\ Spin(9) \\ Spin(10)U(1) \\ Spin(12)SU(2) \\ Spin^{+}(16) \end{gathered}$$

Thank you for your attention!