On de Sitter Space: an Investigation of the Relativity of Inertia and the Cosmological Constant

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An History of Inertia

Einstein's Special Relativity

The Geometry of Riemann

The General Theory of Relativity

Boundary Conditions and the Relativity of Inertia

An Accelerating Universe and de Sitter's Space

Introduction

- The physical laws describing the motion of masses have undergone several major paradigms.
- While there have been large changes, it has always been seen as necessary to include a notion of *inertia*; equivalently a frame of reference in which the laws of physics are simplest.
- We will see that the development of the modern theory of General Relativity was largely informed by the search for an understanding of this idea.

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Galilean Groups and Invariance

- Given a system of objects for whom the position $\vec{x}$ at time t is described consider the following transformations:
  - Translation: $(\vec{x}, t) \mapsto (\vec{x} + \vec{x_0}, t + t_0)$
  - Uniform Motion: $(\vec{x}, t) \mapsto (\vec{x} + t\vec{v_0}, t)$
  - Rotation: $(\vec{x}, t) \mapsto (R\vec{x}, t), \quad R \in O(3)$
- These transformations form a (Lie) group under composition known as the Galilean Tranformations G.
- The Principle of Galilean Relativity asserts that the laws of physics must be invariant under G (Galileo, *Dialogo sopra i due massimi sistemi del mondo*, 1632).

Classical (Newtonian) Mechanics

- Newton's Laws (*Philosophiæ Naturalis Principia Mathematica*, 1687):
  1. An object in motion will tend to stay in motion, unless acted upon by an outside force.
  2. The rate of change of momentum of a body is directly proportional to the forces acting upon it, or $\vec{F} = m\frac{d\vec{v}}{dt}$.
  3. Forces always act in equal and opposite measure.
- The physical laws of Classical Mechanics determined by these laws obey the Principle of Galilean Relativity.

Inertial Frames and the First Law

- The first law asserts that the acceleration of a body acted upon by a net zero force vanishes.
- However, we can only apply this to the motion of a body as described in an inertial reference frame; that is a frame that is itself not undergoing acceleration.
- It is not clear how to determine, a priori, which frames are inertial.
- One approach is to observe that in non-inertial reference frames "fictitious forces" without apparent source will appear, but it's not self-evident which forces should be considered fictitious.

Absolute Frame of Rest

At this point (prior to ∼1900) in the development of physics there were a few implicitly accepted notions about inertia:

- There must be a distinguished Absolute Frame of Rest, and inertial frames can be defined as those that are not accelerating with respect to it.
- No mechanical experiment can determine which frames are inertial.
- The Principle of Galilean Relativity is convenient,but simply a happy coincidence.

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Modern Problems

- Following the work on electromagnetic theory by Maxwell, Gauss, Faraday, Ampére, Boltzmann, and others in the late 1800's, there was a crisis:
  - The Principle of Galilean Relativity asserts that observers in different inertial frames should measure different velocities for the speed of light.
  - Maxwell's Equations assert that the speed of light must always be the same value c for any observer.
- Coupled with the problem of the Relativity of Inertia and some measured inconsistencies with experiment, there was growing consensus that Classical Mechanics was incomplete.

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Special Relativity

- In "Zur Elektrodynamik bewegter Körper" (1905) Einstein proposes a new system of mechanics.
- In particular, he asserts that correct physical laws must be invariant under the Lorentz Transformations:
  - Translation: $(\vec{x}, t) \mapsto (\vec{x} + \vec{x_0}, t + t_0)$
  - Boost:

$$(\vec{x}, t) \mapsto (\vec{x}_\perp, 0) + \frac{1}{\sqrt{1 - \left(\frac{||\vec{v_0}||}{c}\right)^2}} \left(\vec{x}_{||} - t\vec{v_0}, t - \frac{\vec{v_0} \cdot \vec{x}}{c^2}\right)$$

    where $\vec{x}_{||} = \mathrm{proj}_{\vec{v_0}} \vec{x}, \quad \vec{x}_\perp = \vec{x} - \vec{x}_{||}$.
  - Rotation: (omitted for reasons of space)

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Minkowski spacetime

- Newton's second law

$$\vec{F} = \frac{d\vec{x}}{dt}$$

is inconsistent with Lorentz invariance.

- To resolve this, the differential equations relating force to acceleration in Special Relativity become

$$\vec{F} = \frac{d\vec{p}}{dt}$$

where $\vec{p}(t)$ describes position in Minkowski spacetime $R^{1,3}$.

- This is topologically R^4, equipped with the inner product of vectors

$$(t_1, x_1, y_1, z_1) \cdot (t_2, x_2, y_2, z_2) = t_1 t_2 - x_1 x_2 - y_1 y_2 - z_1 z_2$$

Relativity of Inertia

- There is, still, an essential unanswered question that carries over from the Classical Mechanics:
  How can one determine which frames are inertial?
- Principle of (Ernst) Mach: "Mass out there influences inertia here."
- "You are standing in a field looking at the stars. Your arms are resting freely at your side, and you see that the distant stars are not moving. Now start spinning. The stars are whirling around you and your arms are pulled away from your body. Why should your arms be pulled away when the stars are whirling? Why should they be dangling freely when the stars don't move?"
  - Steven Weinberg, *Gravitation and Cosmology* (1972)

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Topological Manifolds

We will need a brief seque into Riemannian Geometry.

- Let $(\mathbb{M}, \mathcal{T})$ be a topological space that has the following properties:
  - Hausdorff: There are enough open sets, so it is possible to distinguish between any two points.
  - Second-Countable: There aren't too many open sets, so the space can have interesting structure.
  - Locally Euclidean: Every point has a neighborhood that is homeomorphic to $\mathbb{R}^n$.
- $(\mathbb{M}, \mathcal{T})$ is called a topological manifold.

Smooth Manifolds

In order to have a notion of derivative, we need to guarantee that it is well defined.

- Let $U, V \in \mathcal{T}$ be homeomorphic to $\mathbb{R}^n$ by maps ϕ_U, ϕ_V, and suppose $U \cap V \neq \emptyset$; denote $E = U \cap V$.
- We call the continuous map

$$\phi_U|_E \circ \phi_V^{-1}|_{\phi_V(E)} \colon \phi_V(E) \to \phi_U(E)$$

  a transition map.

Definition

If there is a collection $\mathcal{F}$ of smooth (infinitely differentiable) transition maps that cover $\mathbb{M}$, we call $(\mathbb{M}, \mathcal{T}, \mathcal{F})$ a smooth manifold.

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The Tangent Bundle

We will require a notion of vector fields.

- Fix a point $p \in \mathbb{M}^n$; at this point "attach" a copy of $\mathbb{R}^n$ at the origin (think tangent planes to surfaces). We call these tangent spaces $T_p\mathbb{M}$, and their disjoint union the tangent bundle $T\mathbb{M} = \cup_{p \in \mathbb{M}} T_p\mathbb{M}$.
- The vectors in $T_p\mathbb{M}$ can be put into 1-1 correspondence with the smooth curves $\gamma\colon [0,1] \to \mathbb{M}$ passing through p "in the same direction", so we view vectors in $T_p\mathbb{M}$ as tangent vectors to curves.
- If at every point p in $\mathbb{M}$ we choose a tangent vector X_p, we call the collection a vector field X.

Riemannian Manifolds

At this point, we're equipped to generalize the Euclidean dot product.

- Recall that on $\mathbb{R}^n$, for two vectors $\vec{v}, \vec{u}$ it holds that $\vec{u} \cdot \vec{v} = \|u\|\|v\| \cos(\theta)$
- To recover this, we will introduce a (positive-definite) symmetric bilinear map

$$g_p(\cdot, \cdot) \colon T_p\mathbb{M} \times T_p\mathbb{M} \to \mathbb{R}$$

  on every tangent space $T_p\mathbb{M}$.
- If these inner products vary smoothly as a function of p, we call it a (pseudo-)Riemannian metric denoted g.
- A smooth manifold with a (pseudo-)Riemannian metric $(\mathbb{M}, \mathbb{T}, \mathcal{F}, g)$ is called a (pseudo-)Riemannian manifold.

Distance as Arclength

Now we can make sense of length and distance.

- The magnitude of a tangent vector X_p at $p \in \mathbb{M}$ is given by the usual inner product formula

$$\|X_p\| = \sqrt{g_p(X_p, X_p)}$$

- The length of a curve $\gamma \colon [0,1] \to \mathbb{M}$ is given by the usual arclength formula computed using its velocity vectors $\dot{\gamma}$

$$L(\gamma) = \int_0^1 \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}\; dt$$

Connections

We want to generalize the gradient of a function ∇f to an operator that makes sense on vector fields.
Denoting the space of vector fields $\mathfrak{X}(\mathbb{M})$, we want an operator

$$\nabla \colon \mathfrak{X}(\mathbb{M}) \times \mathfrak{X}(\mathbb{M}) \to \mathfrak{X}(\mathbb{M})$$

such that

1. $\nabla_{fX+Y} Z = f\nabla_X Z + \nabla_Y Z$
2. $\nabla_X (fY) = df(X)Y + f\nabla_X Y$

where $df(X) = g(\nabla f, X)$.

Any such ∇ is called a connection.

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Levi-Civita Connection

While there are many possible connections on a manifold, there is one in particular that we are interested in.

Theorem (Fundamental Theorem of Riemannian Geometry)

There exists a unique connection ∇ such that

1. $\nabla_X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$
2. $\nabla_X Y - \nabla_Y X = [X, Y]$

This connection is called the Levi-Civita connection.

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Parallel Transport

Equipped with a connection, we can define the notion of parallel transport.

- Let γ be a curve in $\mathbb{M}$. We say that a vector field X is parallel along γ if
$$\nabla_{\dot{\gamma}} X = 0$$
- Given a curve γ from p to q and a vector $u \in T_p\mathbb{M}$, we will call a vector $v \in T_q\mathbb{M}$ the parallel transport of u along γ if there is a γ-parallel vector field X such that $X_p = u, X_q = v$.

Geodesics

In particular, we can generalize to this setting a notion of "straight" path.

- Suppose γ is a curve such that its tangent vector field $\dot{\gamma}$ is parallel. That is

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0.$$

  We will call such a curve geodesic.

- On Riemannian manifolds it can be shown that around every point $p \in \mathbb{M}$ there exists a neighborhood U_p such that there exists a unique length-minimizing curve between p and any $q \in U_q$; these curves will always be geodesics for the Levi-Civita connection.

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Curvature

We can also define curvature using connections.

1. Riemannian Curvature:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{\nabla_X Y - \nabla_Y X} Z$$

2. Ricci Curvature:

$$\mathrm{Ric}(X,Y) = Tr_g(Z \mapsto R(Z,X)Y)$$

3. Scalar Curvature:

$$\mathrm{S} = Tr_g(Ric)$$

- These are invariant by choice of coordinates, and measure the noncommutativity of second derivatives.
- They uniformly vanish on $\mathbb{R}^n$.

Mach and Riemann

Einstein, seeking to address outstanding issues with Special Relativity, determined to generalize to a model of the universe as a 4-dimensional pseudo-Riemannian manifold with metric signature $(1, 3)$.

- He insisted that the following must hold:
  1. The Principle of Lorentzian Relativity: The laws of physics must be equivalent in all admissible frames and invariant by Lorentz Transformations.
  2. The Principle of Equivalence: That in the limit the laws of Classical Mechanics must be recovered.
  3. The Principle of Mach: That the metric must be "completely determined by the mass of bodies."
- In regards to the third, which contemporary physicists were slow to adopt, he said "This contentedness [to go without Mach's principle] will appear incomprehensible to a later generation however."

General Relativity

In "Die Feldgleichungen der Gravitation" (1915) Einstein proposes his theory of General Relativity.

- First, it is asserted that one can determine a $(4,0)$ stress-energy tensor $\mathbb{T}$ from the distribution of mass in spacetime.
- Then this tensor is claimed to be proportional to the Einstein tensor G determining the curvature and therefore the pseudo-Riemannian metric of spacetime. In particular,

$$\mathbb{G} = \kappa \mathbb{T}$$

Einstein Tensor

It isn't immediately clear how to make an appropriate choice for $\mathbb{G}$.

- He insisted that $\mathbb{T}$ must be divergence-free, in analogy with fluid mechanics.
- Notably, the Riemann curvature tensor (a natural choice) R does not satisfy this.
- Instead, the curvature quantity

$$\mathbb{G} = \mathsf{Ric} - \mathsf{S}\, g$$

is proposed; it is divergence free and introduces the metric in a natural way.

Einstein's Field Equations

- With this, the Einstein Field Equations become

$$\mathrm{Ric} - \mathrm{S}\, g = \frac{8\pi G}{c^2}\mathbb{T}$$

- The proportionality constant $\kappa = \frac{8\pi G}{c^2}$ is determined by an appeal to the Principle of Equivalence.

A Resolution to the Problem of Inertia?

With the introduction of General Relativity, Einstein hoped to resolve the problem of Relativity of Inertia.

- Inertial reference frames are defined as those which move along geodesics!
- We have the following situation:
  - The Einstein Field Equations are a system of 10 simultaneous, nonlinear PDEs that determine the metric.
  - Masses will move along geodesic paths unless acted upon by external forces, in which case $\vec{F} = \frac{d\vec{p}}{dt}$.
  - The motion of masses then changes the form of the stress-energy tensor $\mathbb{T}$ which correspondingly changes the metric.
  - We thereby determine a highly nonlinear dynamical system describing the motion of bodies.

Boundary conditions at Infinity

There remains, however, a gap.

- In order to solve Einstein's field equations, one must provide a boundary condition for the system.
- Clearly the Principle of Equivalence implies that in some neighborhood of the Earth the metric should approximate that of Minkowski space

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

- However, as the distance from our neighborhood increases to infinity it is unclear what the appropriate choice of boundary conditions is.

Static Universe

In fact, the natural assumption that the initial values of the metric remain Minkowski imply that space collapses in finite time.

- Dissatisfied with this, holding the expectation that the universe should be "static and eternal," Einstein in "Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie" (1917) introduces the Cosmological Constant Λ to the field equations, which become

$$\mathrm{Ric} + \left(\Lambda - \frac{1}{2}\mathrm{S}\right) g = \frac{8\pi G}{c^2}\mathbb{T}$$

- For $\Lambda > 0$ a topologically spherical, static universe is possible.

Enter Willem de Sitter

In 1917 Willem de Sitter publishes "On the Relativity of Inertia."

- His purpose is to examine the implications of Einstein's modifications to the theory, and argue against some uncomfortable consequences.
- In particular, he compares two solutions of the field equations with $\Lambda > 0$; the system (A) proposed by Einstein, in which the metric vanishes in the spacial directions at infinity, and an alternative system (B), in which the metric vanishes identically.
- That is, at an infinite distance from the origin,

$$g^{(A)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad g^{(B)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Hyperspherical coordinates and Computation of the metric

In his analysis, de Sitter establishes a coordinate system locally by applying sterographic projection.

- The metrics can be computed as

$$g_{ij}^{(A)} = -\frac{\delta_{ij}}{(1+\sigma r^2)^2}, \qquad g_{11}^{(A)} = 1,$$
$$g_{ij}^{(B)} = -\frac{\delta_{ij}}{(1+\sigma h^2)^2}, \qquad g_{11}^{(B)} = \frac{\delta_{ij}}{(1+\sigma h^2)^2}$$

- We use radial coordinates $r^2 = x^2 + y^2 + z^2$ and $h^2 = x^2 + y^2 + z^2 - (ct)^2$.
- The quantity

$$\sigma = \frac{1}{4R^2}$$

is determined by the the radius R of the hypersphere.

Solutions

From here he can solve the field equations.

- In order to find all "reasonable" solutions de Sitter assumes that there can exist matter of constant density ρ throughout the universe, in which case the energy-stress tensor takes the form $T_{11} = g_{11}\rho$ and the rest of its components vanish.
- The field equations become

$$\mathbb{G}_{ij} - (\Lambda + \frac{1}{2}\kappa\rho)g_{ij} = 0$$
$$\mathbb{G}_{11} - (\Lambda + \frac{1}{2}\kappa\rho)g_{11} = -\kappa\rho$$

- With boundary conditions (A) and (B) it is found that

$$\Lambda_{(A)} = 4\sigma, \rho_{(A)} = \frac{8\sigma}{\kappa}$$
$$\Lambda_{(B)} = 12\sigma, \rho_{(B)} = 0$$

World-Matter

- As a consequence, the Einstein universe (A) necessarily imply that the universe is full of a constant density matter, which de Sitter calls the world-matter.
- de Sitter's Universe (B) does not suffer from this uncomfortable demand.
- Notably, either system finally gives a resolution to the Relativity of Inertia.

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de Sitter Space

- de Sitter also observed that in his system (B) the metric

$$g = g^{(B)} = \frac{1}{(1+\sigma h^2)^2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

explodes to $+\infty$ at

$$0 = 1 + \sigma h^2 \implies 4R^2 = c^2 t^2 - x^2 - y^2 - z^2$$

- This implies then that the de Sitter universe necessarily has the topology of a hyperboloid!

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On the Curvature of Space

In 1918, amid ongoing debate with Einstein about the systems described in the previous paper, de Sitter publishes "On the Curvature of Space".

- The purpose of the paper is to investigate further the consequences of the Einstein and de Sitter universes.
- In particular, de Sitter begins by arguing that a perfectly (hyper)spherical model is undesirable, as it implies the geodesic lines (and thus rays of light) emitted from any point will coalesce at the antipodal point; this is avoided by instead assuming ellipticity.

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Considerations of Light Rays

Being massless, light rays always follow the geodesics $\|\dot{\gamma}\|^2 = 0$.

- de Sitter observes that these are linear in hyperspherical coordinates for Einstein's universe, while they are linear in hyperbolical (sic) coordinates for his. This further reinforces that he has arrived at a unique topology for the universe.
- Since geodesic lines are closed in these models, it should be expected that light would travel the long "voyage around the universe", for example the sun should be observable on both sides of the Earth.
  - In Einstein's universe this discrepancy can be explained if light is absorbed by the world-matter, but this implies a remarkably small radius R for the universe, on the order of 10^9 AU.
  - This is not a consideration in de Sitter's universe, since the length of geodesics is infinite.

Considerations of Mass

de Sitter also considers the consequences of the presence of matter.

- He studies the system where a “sun” of constant density centered at the origin of his coordinate system.
- It is of particular importance that the theory predict the (very measurable) *motion of the perihelion* of a massive body.
  - In Einstein's universe, the existence of uniform world-matter makes the expression nonintegrable, and so de Sitter carries out a first degree expansion which leads to absurd results. In second degree it gives results consistent with the original theory of General Relativity.
  - In de Sitter's universe the equations are integrable, and also give results that agree.

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Redshift of Nebulae

The spectral lines of astronomical objects can be predicted by knowledge of their composition.

- Deviation from these predictions can only be explained in Classical Mechanics by relative motion (Doppler shift).
- In General Relativity, redshift can occur as a consequence of light traveling near massive bodies, or as a consequence of cosmological expansion.
- One particularly prescient remark by de Sitter is that a systematic observation of redshift of nebulae (and thus cosmological expansion) would be consistent with a de Sitter universe, and not an Einstein universe.

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Later Observations

Later observations did in fact indicate an expanding universe.

- These largely agreed with the predictions of the de Sitter universe made in the 1920s.
- In 1931 Einstein and de Sitter jointly publish "On the Relation between the Expansion and the Mean Density of the Universe."
- Numerous improved and alternate models have been proposed, but the Einstein-de Sitter universe is still fundamental, especially as a model of the early universe during the "inflationary period."
- The positive cosmological constant is a strong motivation for the existence of "dark matter" in the universe, but there is yet to be devised a method of measuring it directly.

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