Totally Geodesic Foliations and sub-Riemannian Geometry

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16 December, 2020

Overview

In this thesis we explore sub-Riemannian structures arising as the transversal distribution to a foliation.

- Connections on foliations
- H-type foliations
- Classification of H-type submersions
- Horizontal Einstein property and GCDI
- Uniform comparison theorems

Foliations

A foliation is a partition of a manifold into equivalence classes that locally models the partition of R^{n+m} by submanifolds R^m.

Definition

Let $\mathbb{M}$ be a $n+m$ dimensional manifold. A foliation is a disjoint collection $\mathcal{F}$ of connected, immersed m-dimensional submanifolds (called leaves) such that for each $p \in \mathbb{M}$ there is a neighborhood U_p and a smooth submersion

$$\phi_{U_p} \colon U_p \to \mathbb{R}^n$$

with the property that for any $x \in \mathbb{R}^n$ the set $f^{-1}(x)$ is either empty or the intersection of one of the submanifolds of $\mathcal{F}$ with U_p.

Foliations

We can see foliations as a local splitting of the tangent bundle

$$T_p\mathbb{M} = \mathcal{H}_p \oplus \mathcal{V}_p$$

where the vertical space $\mathcal{V}_p$ is tangent to the leaf through $p \in \mathbb{M}$. We call the transversal distribution $\mathcal{H}_p$ horizontal.

- It must hold that the Lie bracket

  $$[X, Y] = XY - YX$$

  of two vertical vector fields $X, Y \in \mathcal{V}_p$ remains in $\mathcal{V}_p$; we say that $\mathcal{V}$ is integrable.
- There is no similar restriction on the behavior of $\mathcal{H}$.

Hörmander's condition

Rather, we insist that the horizontal distribution $\mathcal{H}$ be bracket generating; that is, for every $p \in \mathbb{M}$, there exists $n \in \mathbb{N}$ such that

$$T_p\mathbb{M} = \mathrm{Span}\{X_1(p), [X_1(p), X_2(p)], [X_1(p), [X_2(p), X_3(p)]], \\ \dots, [X_1(p), \dots, [X_{n-1}(p), X_n(p)] \dots]\}$$

with $X_1(p), \dots, X_n(p) \in \mathcal{H}_p$.
This is equivalent to Hörmander's condition, and implies that control systems are locally controllable, that SDEs admit smooth densities, and the Chow–Rashevskii theorem:

Theorem (Chow-Rashevskii)

For any $p, q \in \mathbb{M}$, there exists an almost-everywhere horizontal curve γ connecting p to q with finite length.

sub-Riemannian Geometry

Given a smooth manifold $\mathbb{M}$, a bracket-generating distribution $\mathcal{H} \subseteq T\mathbb{M}$, and a fiberwise inner product $g_{\mathcal{H}}$ on $\mathcal{H}$, we say the triple $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$ is a sub-Riemannian manifold.

- We only have a notion of length for horizontal curves almost everywhere tangent to $\mathcal{H}$.
- We can see sub-Riemannian geometry as a constraint on permissible motion.
- These arise naturally in many settings; notably in physics as mechanical problems.

Ricci lower curvature bounds

There are many classical Riemannian results that rely on a Ricci lower curvature bound

$$\mathrm{Ric}(X,X) \geq \kappa g(X,Y)$$

In particular,

- Laplacian (Rauch) Comparison Theorem:

$$\Delta r \leq \begin{cases} (n-1)\sqrt{\kappa}\cot(\sqrt{\kappa}r) & \kappa > 0 \\ \frac{n-1}{r} & \kappa = 0 \\ (n-1)\sqrt{|\kappa|}\coth(\sqrt{|\kappa|}r) & \kappa < 0 \end{cases}$$

- Bonnet-Meyers Diameter Estimates: If $\kappa > 0$ then

$$\mathrm{diam}(\mathbb{M}) \leq \frac{\pi}{\sqrt{\kappa}}$$

and the fundamental group of $\mathbb{M}$ must be finite.

Riemannian Model Spaces

The results in the previous slide follow from comparisons with models spaces; these are the spaces with constant sectional curvature κ.These are precisely:

- $\kappa > 0$, sphere S^n
- $\kappa = 0$, Euclidean space R^n
- $\kappa < 0$, hyperbolic space H^n

each equipped with its canonical Riemannian metric.

Penalty Metric

On a sub-Riemannian manifold $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$ equipped with a Riemannian extension $g = g_{\mathcal{H}} \oplus g_{\mathcal{V}}$, one can consider a penalty metric

$$g_\varepsilon = g_{\mathcal{H}} \oplus \frac{1}{\varepsilon} g_{\mathcal{V}}$$

There is a Gromov-Hausdorff convergence

$$(\mathbb{M}, \mathcal{H}, g_\varepsilon) \xrightarrow{\varepsilon \to 0^+} (\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$$

Unfortunately, the Ricci curvature explodes as

$$\lim_{\varepsilon \to 0^+} \mathrm{Ric}^\varepsilon(X, Y) = \begin{cases} +\infty & X, Y \in \mathcal{V} \\ -\infty & X, Y \in \mathcal{H} \end{cases}$$

Models

There is still hope to understand sub-Riemannian geometry through comparison with models.

- The Heisenberg group is $\mathbb{R}^3$ equipped with vector fields

$$X = \partial_x - \frac{1}{2} y \partial_z, \qquad Y = \partial_y + \frac{1}{2} x \partial_z,$$

  setting $\mathcal{H} = \mathrm{Span}\{X, Y\}$ and defining $g_{\mathcal{H}}$ so that X, Y are orthonormal.
- We can equivalently see this as induced by a submersion

$$\mathbb{R} \hookrightarrow \mathbb{R}^3 \to \mathbb{C}$$

Models

- The Hopf fibration is the sphere S^3 equipped with horizontal distribution induced by the submersion

$$S^1 \hookrightarrow S^3 \to \mathbb{C}P^1$$

- The Anti-de Sitter space is the hyperbolic space H^3 equipped with horizontal distribution induced by the submersion

$$S^1 \hookrightarrow \mathrm{H}^3 \to \mathrm{H}P^1$$

Connections

A map $\nabla\colon \Gamma(T\mathbb{M}) \times \Gamma(T^{i,j}\mathbb{M}) \to \Gamma(T^{i,j}\mathbb{M})$ that is linear in the first component and a derivation in the second component, in analogy with the directional derivative on $\mathbb{R}^n$, is called a connection on $\mathbb{M}$.

Explicitly,

- $\nabla_{fX+Y}U = f\nabla_X U + \nabla_Y U$, and
- $\nabla_X(fU + V) = (Xf)U + f\nabla_X U + \nabla_X V$.

Levi-Civita Connection

There exists on any Riemannian manifold $(\mathbb{M}, g)$ a connection ∇^g uniquely defined by the properties

- ∇^g is metric, that is
$$\nabla^g g = 0$$
- ∇^g is torsion-free, that is
$$T(X, Y) = \nabla^g_X Y - \nabla^g_Y X - [X, Y] = 0$$

This is called the Levi-Civita connection.

Koszul Formula

For any metric connection ∇, it can be shown that

$$\nabla_X Y - \nabla^g_X Y = A(X, Y)$$

where we define

$$2g(A(X, Y), Z) = g(T^\nabla(X, Y), Z) + g(T^\nabla(Z, X), Y) - g(T^\nabla(Y, Z), X).$$

It follows that any metric connection is uniquely determined by a formula for A independent of ∇.

Adapted connections on foliations

It is of interest to understand how a connection ∇ will interact with the structure of a foliation.
In particular,

Definition

Let $(\mathbb{M}, \mathcal{F})$ be a foliation with vertical distribution $\mathcal{V}$ and transversal distribution $\mathcal{H}$. If ∇ is a connection on $\mathbb{M}$ such that

- $\nabla_X Y \in \Gamma(\mathcal{H})$ for all $Y \in \Gamma(\mathcal{H})$, and
- $\nabla_X Z \in \Gamma(\mathcal{V})$ for all $Z \in \Gamma(\mathcal{V})$,

we say ∇ is adapted to the foliation.

Riemannian foliations

We have a few important definitions for manifolds equipped with both a foliation and Riemannian structure.

Definition

Suppose $(\mathbb{M}, g)$ is a Riemannian manifold and $(\mathbb{M}, \mathcal{F})$ is a foliation.

- If the metric splits orthogonally as $g = g_{\mathcal{H}} \oplus g_{\mathcal{V}}$ we say $(\mathbb{M}, g, \mathcal{F})$ is a Riemannian foliation.
- If the leaves are totally-immersed submanifolds, we say the foliation is totally geodesic.
- If the local submersions are diffeomorphisms we say the metric is bundle-like.

Bott Connection on foliations

Given a totally-geodesic foliation $(\mathbb{M}, g, \mathcal{F})$ we can define the Bott connection ∇^B

$$\nabla^B_X Y = \begin{cases} \mathrm{pr}_{\mathcal{H}} \nabla^g_X Y & X, Y \in \mathcal{H} \\ \mathrm{pr}_{\mathcal{H}}[X, Y] & X \in \mathcal{V}, Y \in \mathcal{H} \\ \mathrm{pr}_{\mathcal{V}}[X, Y] & X \in \mathcal{H}, Y \in \mathcal{V} \\ \mathrm{pr}_{\mathcal{V}} \nabla^g_X Y & X, Y \in \mathcal{V} \end{cases}$$

- The Bott connection is metric, but has nonvanishing torsion

$$T^B(X, Y) = -\mathrm{pr}_{\mathcal{V}}[\mathrm{pr}_{\mathcal{H}} X, \mathrm{pr}_{\mathcal{H}} Y]$$

- Importantly, the Bott connection is adapted to the foliation.

Foliations inducing sR geometry

As the models indicate, many sub-Riemannian manifolds arise from foliations.

- Given a Riemannian manifold $\mathbb{M}$ foliated with totally-geodesic leaves $\mathcal{V}$, a choice of transversal bracket-generating distribution $\mathcal{H}$ that splits the metric orthogonally as $g = g_{\mathcal{H}} \oplus g_{\mathcal{V}}$ will give a sub-Riemannian structure $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$.
- While the vertical space $\mathcal{V}$ is not intrinsic to the sub-Riemannian structure, its properties have consequences for the sub-Riemannian structure.

H-type foliations

Given a totally-geodesic foliation with complement $(\mathbb{M}, \mathcal{V}, \mathcal{H}, g)$ and the associated Bott connection ∇^B, one can define for each $Z \in \mathcal{V}$ an endomorphism J_Z of $\mathcal{H}$ by

$$g(J_Z X, Y) = g(T^B(X, Y), Z)$$

Definition

If for all $Z \in \mathcal{V}$, $X, Y \in \mathcal{H}$ it holds that

$$g(J_Z X, J_Z Y) = \|Z\|^2 g(X, Y)$$

we say $(\mathbb{M}, \mathcal{H}, g)$ is an H-type foliation.

Covariant derivatives of T^B

We classify H-type foliations by the behavior of the covariant derivatives of the torsion:

- We say it is Yang-Mills if the horizontal divergence

$$\delta_{\mathcal{H}} T^B = \mathrm{Tr}_{\mathcal{H}}(\nabla^B_{\times} T^B)(\times, \cdot)$$

  vanishes. This always holds, and implies a generalized curvature dimension inequality.

- We say it has horizontally parallel torsion if

$$\nabla^B_{\mathcal{H}} T^B = 0$$

- We say it has completely parallel torsion if

$$\nabla^B T^B = 0$$

Examples

Structure	Torsion
Complex Type, $m = 1, n = 2k$	
K-Contact Manifolds	YM
Heisenberg Group, Hopf, Anti de-Sitter Fibrations	CP
Twistor Type, $m = 2, n = 4k$	
Twistor space over quaternionic Kähler manifold	HP
Projective Twistor space $\mathbb{CP}^1 \hookrightarrow \mathbb{C}P^{2k+1} \to \mathbb{H}P^k$	HP
Hyperbolic Twistor space $\mathbb{CP}^1 \hookrightarrow \mathbb{C}H^{2k+1} \to \mathbb{H}H^k$	HP
Quaternionic Type, $m = 3, n = 4k$	
3-Sasakian Manifolds	HP
Torus bundle over hyperkähler manifolds	CP
Quaternionic Heisenberg Group, Hopf, and Anti-de Sitter Fibrations	CP/HP
Octonionic Type, $m = 7, n = 8$	
Octonionic Heisenberg Group, Hopf, and Anti-de Sitter Fibrations	CP/HP
H-type Groups, m is arbitrary	CP

Curvature Dimension Inequalities

There is the notion of Curvature Dimension Inequality

$$\|\nabla^2 f\|^2 + \mathrm{Ric}(\nabla f, \nabla f) \geq \frac{1}{n}(\Delta f)^2 + \rho\|\nabla f\|^2$$

- This is known to be equivalent on a Riemannian manifold to a Ricci lower curvature bound.
- Interestingly, many of the Riemannian results of interest that classically follow from Ricci lower curvature bounds can be proved directly from this inequality.

Generalized Curvature Dimension Inequality

The CDI cannot hope to hold on sub-Riemannian spaces because of the explosion of the Ricci curvature; the Generalized Curvature Dimension Inequality

$$\|\nabla^2_{\mathcal{H}} f\|^2 + \nu \|\nabla^2_{\mathcal{V}} f\|^2 \geq \frac{1}{n}(\Delta_{\mathcal{H}} f)^2 + \left(\rho_1 - \frac{\kappa}{\nu}\right) \|\nabla_{\mathcal{H}} f\|^2 + \rho_2 \|\nabla_{\mathcal{V}} f\|^2$$

was introduced by Baudoin and Garofalo to address precisely this pathology.

- The Yang-Mills property with a horizontal Ricci lower curvature bound is sufficient to imply the GCDI.
- From this, we can recover several classical results.

Clifford Structures

The relation

$$J_{Z_1} J_{Z_2} + J_{Z_2} J_{Z_1} = -2g(Z_1, Z_2)\,\mathrm{Id}$$

holds, which implies that we can extend the J map to the Clifford algebra $Cl(\mathcal{V})$.

- We derive a classification of H-type submersions with <u>horizontally parallel Clifford structure</u> $\Psi \colon \mathcal{V} \times \mathcal{V} \to Cl_2(\mathcal{V})$

$$(\nabla^B_W J)_Z = J_{\Psi(W,Z)}$$

- It must hold that for some $\kappa \in \mathbb{R}$,

$$\Psi(u, v) = -\kappa(u \cdot v + g(u, v))$$

Classification of H-type submersions with horizontally parallel Clifford structure, $\kappa > 0$

$\mathbb{M}$	$\mathbb{B}$	Fiber	rank($\mathcal{H}$)	rank($\mathcal{V}$)
Twistor space	Quaternion-Kähler with positive scalar curvature	$\mathbb{S}^2$	$4k$	2
3-Sasakian	Quaternion-Kähler with positive scalar curvature	$\mathbb{S}^3$	$4k$	3
Quaternion-Sasakian	Product of two quaternion-Kähler with positive scalar curvature	$\mathbb{R}P^3$	$4k$	3
$\frac{\mathrm{Sp}(q^+ +1)\times\mathrm{Sp}(q^- +1)}{\mathrm{Sp}(q^+)\times\mathrm{Sp}(q^-)\times\mathrm{Sp}(1)}$	$\mathbb{H}P^{q^+}\times\mathbb{H}P^{q^-}$	$\mathbb{S}^3$	$4(q^+ + q^-)$	3
$\frac{\mathrm{Sp}(k+2)}{\mathrm{Sp}(k)\times\mathrm{Spin}(4)}$	$\frac{\mathrm{Sp}(k+2)}{\mathrm{Sp}(k)\times\mathrm{Sp}(2)}$	$\mathbb{S}^4$	$8k$	4
$\frac{\mathrm{SU}(k+4)}{\mathrm{S}(\mathrm{U}(k)\times\mathrm{Sp}(2)\mathrm{U}(1))}$	$\frac{\mathrm{SU}(k+4)}{\mathrm{S}(\mathrm{U}(k)\times\mathrm{U}(4))}$	$\mathbb{R}P^5$	$8k$	5
$\frac{\mathrm{SO}(k+8)}{\mathrm{SO}(k)\times\mathrm{Spin}(7)}$	$\frac{\mathrm{SO}(k+8)}{\mathrm{SO}(k)\times\mathrm{SO}(8)}$	$\mathbb{R}P^7$	$8k$, $k\geq 3$, k odd	7
$\frac{\mathrm{Spin}(k+8)}{\mathrm{SO}(k)\times\mathrm{Spin}(7)}$	$\frac{\mathrm{SO}(k+8)}{\mathrm{SO}(k)\times\mathrm{SO}(8)}$	$\mathbb{S}^7$	$8k$, $k=1$, k even	7
Exceptional cases				
$\frac{F_4}{\mathrm{Spin}(8)}$	$\frac{F_4}{\mathrm{Spin}(9)} = \mathbb{O}P^2$	$\mathbb{S}^8$	16	8
$\frac{E_6}{\mathrm{Spin}(8)\mathrm{U}(1)}$	$\frac{E_6}{\mathrm{Spin}(10)\mathrm{U}(1)} = (\mathbb{C}\otimes\mathbb{O})P^2$	$\mathbb{S}^9$	32	9
$\frac{E_7}{\mathrm{Spin}(11)\mathrm{SU}(2)}$	$\frac{E_7}{\mathrm{Spin}(12)\mathrm{SU}(2)} = (\mathbb{H}\otimes\mathbb{O})P^2$	$\mathbb{S}^{11}$	64	11
$\frac{E_8}{\mathrm{Spin}(15)}$	$\frac{E_8}{\mathrm{Spin}^+(16)} = (\mathbb{O}\otimes\mathbb{O})P^2$	$\mathbb{S}^{15}$	128	15

Diameter and first eigenvalue estimates

The horizontally parallel Clifford structure moreover implies a horizontal Einstein condition

$$\mathrm{Ric}_{\mathcal{H}}(X, Y) = \kappa g_{\mathcal{H}}(X, Y)$$

Applying the GCDI, we recover on H-type foliations with horizontally parallel Clifford structure:

- Bonnet-Myers type diameter bounds
- Lower bounds on the first eigenvalue for the sub-Laplacian

These results are purely sub-Riemannian, in the sense that they are independent of a choice of $\mathcal{V}$.

Comparison theorems in the limit

We now consider a different approach, recalling the penalty metric

$$g = g_{\mathcal{H}} \oplus \frac{1}{\varepsilon} g_{\mathcal{V}}.$$

- Fix $p \in \mathbb{M}$, and define

$$r_\varepsilon(q) = d_\varepsilon(p,q) = \inf_{\gamma \in C(p,q)} \int_\gamma \|\nabla \gamma(t)\|_\varepsilon dt.$$

  Where $C(p,q)$ is the collection of curves connecting p to q.
- On compact sets, we have uniform convergence

$$d_\varepsilon(p,q) \xrightarrow{\varepsilon \to 0^+} d_{cc}(p,q)$$

Metric Connections, Jacobi Equation

We proceed by recovering a Jacobi equation for variations of geodesics.

- The Bott connection will no longer suffice since its adjoint

$$\hat{\nabla}^B = \nabla^B + T^B$$

 is not metric.

- We introduce for any $\varepsilon > 0$ the metric connection with metric adjoint

$$\hat{\nabla}^{\varepsilon}_X Y = \nabla_X Y + \frac{1}{\varepsilon} J_X Y.$$

- For a g_ε-geodesic γ, the Jacobi equation in this setting is

$$\hat{\nabla}^{\varepsilon}_{\dot\gamma} \nabla^{\varepsilon}_{\dot\gamma} W + \hat{R}^{\varepsilon}(W, \dot\gamma)\dot\gamma = 0$$

The Comparison Principle

Theorem (Baudoin, Grong, Kuwada, & Thalmaier '17)

- *Let* $x, y \in \mathbb{M}$*,*
- $\gamma\colon [0, r_\varepsilon] \to \mathbb{M}$ *a unit speed* g_ε*-geodesic connecting* x, y*, and*
- $W_1, \cdots, W_k$ *be a collection of vector fields along* γ *such that*

$$\sum_{i=0}^{k} \int_0^{r_\varepsilon} \langle \hat{\nabla}^\varepsilon_{\dot\gamma} \nabla^\varepsilon_{\dot\gamma} W_i + \hat{R}^\varepsilon(W_i, \dot\gamma)\dot\gamma, W_i \rangle_\varepsilon \geq 0$$

then at $y = \gamma(r_\varepsilon)$ *it holds that*

$$\sum_{i=0}^{k} \mathrm{Hess}^{\hat{\nabla}^\varepsilon}(r_\varepsilon)(W_i, W_i) \leq \sum_{i=0}^{k} \langle W_i, \hat{\nabla}^\varepsilon_{\dot\gamma} W_i \rangle_\varepsilon$$

with equality if and only if the W_i *are Jacobi fields.*

Horizontal Splitting

We introduce an orthogonal splitting of the horizontal bundle. Fixing a vector field $Y \in \mathcal{H}$,

$$\mathcal{H} = \operatorname{span}(Y) \oplus \mathcal{H}_{Riem}(Y) \oplus \mathcal{H}_{Sas}(Y)$$

where

$$\mathcal{H}_{Sas}(Y) = \{J_Z Y | Z \in \mathcal{V}\}$$

$$\mathcal{H}_{Riem}(Y) = \{X \in \mathcal{H} | X \perp \mathcal{H}_{Sas} \oplus \operatorname{span}(Y)\}$$

Lemma

Denoting $n = \operatorname{rk}(\mathcal{H}), m = \operatorname{rk}(\mathcal{V})$, *we will have*

$$\dim(\mathcal{H}_{Sas}) = m, \quad \dim(\mathcal{H}_{Riem}) = n - m - 1$$

Comparison Functions

Simliarly to the Riemannian case, we consider the comparision functions

$$F_{Riem}(r,\kappa) = \begin{cases} \sqrt{\kappa}\cot(\sqrt{\kappa} r) & \text{if } \kappa > 0 \\ \frac{1}{r} & \text{if } \kappa = 0 \\ \sqrt{|\kappa|}\coth(\sqrt{|\kappa|} r) & \text{if } \kappa < 0 \end{cases}$$

$$F_{Sas}(r,\kappa) = \begin{cases} \frac{\sqrt{\kappa}\left(\sin(\sqrt{\kappa} r) - \sqrt{\kappa} r\cos(\sqrt{\kappa} r)\right)}{2-2\cos(\sqrt{\kappa} r) - \sqrt{\kappa} r\sin(\sqrt{\kappa} r)} & \text{if } \kappa > 0 \\ \frac{4}{r} & \text{if } \kappa = 0 \\ \frac{\sqrt{\kappa}\left(\sqrt{\kappa} r\cosh(\sqrt{\kappa} r) - \sinh(\sqrt{\kappa} r)\right)}{2-2\cosh(\sqrt{\kappa} r) + \sqrt{\kappa} r\sinh(\sqrt{\kappa} r)} & \text{if } \kappa < 0 \end{cases}$$

These comparison functions will correspond to the splitting of $\mathcal{H}$.

Hessian Comparisons

Theorem (Baudoin, Grong, Rizzi, & M. '19)

- *Let* $\gamma \colon [0, r_\varepsilon] \to \mathbb{M}$ *be a* g_ε*-geodesic. Then*

$$\mathrm{Hess}(r_\varepsilon)(\dot\gamma, \dot\gamma) \leq \frac{\|\dot\gamma\|^2 \left(1 - \|\dot\gamma\|^2\right)}{r_\varepsilon}$$

- *If* $\mathrm{Sec}(X \wedge Y) \geq \rho > 0$ *for all unit* $X, Y \in \mathcal{H}_{Riem}(\dot\gamma)$*, then*

$$\mathrm{Hess}(r_\varepsilon)(X, X) \leq F_{Riem}(r_\varepsilon, K)$$

- *If* $\mathrm{Sec}(X \wedge J_Z X) \geq \rho > 0$ *for all unit* $X \in \mathcal{H}_{Sas}(\dot\gamma)$*, then*

$$\mathrm{Hess}(r_\varepsilon)(X, X) \leq F_{Sas}(r_\varepsilon, K)$$

Where K *is a constant depending on* $\rho, \varepsilon, \|\nabla_{\mathcal{V}} r_\varepsilon\|$*, and* $\|\nabla_{\mathcal{H}} r_\varepsilon\|$*.*

Proof Sketch: Sasakian Hessian Comparison

Let's consider the Sasakian case.

- Fix $p, q \notin \mathrm{Cut}_\varepsilon(p)$ and let γ be the length-minimizing geodesic connecting p to q.
- Let $X \in \mathcal{H}_{Sas}(\dot\gamma)$; then there exists some $Z \in \Gamma(\mathcal{V})$ such that

$$X = J_Z\dot\gamma$$

- Define $Z_\perp = Z - g_\varepsilon(Z, \dot\gamma)\dot\gamma$, and let

$$W(t) = a(t)J_Z\dot\gamma + b(t)Z_\perp$$

for some undetermined functions $a(t), b(t)$.

Proof Sketch: Sasakian Hessian Comparison

- We set initial conditions $a(0) = b(0) = b(r_\varepsilon) = 0, a(r_\varepsilon) = 1$.
- Assuming constant sectional curvature ρ and setting $K = \rho\|\nabla_{\mathcal{H}} r_\varepsilon\|^2 + \|\nabla_{\mathcal{V}} r_\varepsilon\|^2$, we find that $W(t)$ will be a Jacobi field if and only if

$$\begin{aligned} \ddot{a}\varepsilon + \dot{b} + a(\varepsilon K - 1) &= 0 \\ \ddot{b} - \dot{a} &= 0 \end{aligned}$$

- Explicitly, the general solution for $r_\varepsilon < \frac{1}{2\pi}\sqrt{K}$ is

$$\begin{aligned} a(t) &= C_1 \cos(\sqrt{K}t) + C_2 \sin(\sqrt{K}t) + C_3 \\ b(t) &= C_1 \frac{\sin(\sqrt{K}t)}{\sqrt{K}} + C_2 \frac{1 - \cos(\sqrt{K}t)}{\sqrt{K}} - C_3 \varepsilon K t + C_4 \end{aligned}$$

Proof Sketch: Sasakian Hessian Comparison

- Applying the Comparison Principle,

$$\mathrm{Hess}(r_\varepsilon)(X,X) \leq \sqrt{K}\frac{\sin(r_\varepsilon\sqrt{K}) - (1-\varepsilon K)r_\varepsilon\sqrt{K}\cos(r_\varepsilon\sqrt{K})}{2-2\cos(r_\varepsilon\sqrt{K}) - (1-\varepsilon K)r_\varepsilon\sqrt{K}\sin(r_\varepsilon\sqrt{K})}$$

- Observing that the above expression always has negative first derivative with respect to ε, we conclude

$$\mathrm{Hess}(r_\varepsilon)(X,X) \leq F_{Sas}(r_\varepsilon, K)$$

Diameter Estimates

Theorem (Baudoin, Grong, M., & Rizzi, '19)

Let $\kappa > 0$. Then for unit $X \in \mathcal{H}$,

1. $$\frac{\mathrm{Ric}_{Riem}(X,X)}{n-m-1} \geq \kappa \implies \mathrm{diam}_0(\mathbb{M}) \leq \frac{\pi}{\sqrt{\kappa}}$$
2. $$\mathrm{Sec}(X \wedge J_Z X) \geq \kappa \implies \mathrm{diam}_0(\mathbb{M}) \leq \frac{2\pi}{\sqrt{\kappa}}$$
3. $$\frac{\mathrm{Ric}_{Sas}(X,X)}{m} \geq \kappa \implies \mathrm{diam}_0(\mathbb{M}) \leq \frac{2\pi\sqrt{3}}{\sqrt{\kappa}}$$

and in each case the fundamental group of $\mathbb{M}$ must be finite.

The first two of these are sharp, as they are achieved in the complex, quaternionic, and octonionic Hopf fibrations.

sub-Laplacian

Similarly to the horizontal Ricci curvature, we can define the sub-Laplacian as the trace of the Hessian. For the distance function r_ε along a geodesic γ with $Y = \nabla_{\mathcal{H}} r_\varepsilon$,

$$\Delta_{\mathcal{H}} r_\varepsilon = \sum_{i=0}^{n} \mathrm{Hess}(r_\varepsilon)(W_i, W_i)$$

$$= \mathrm{Hess}(r_\varepsilon)(Y, Y) + \sum_{i=0}^{m} \mathrm{Hess}(r_\varepsilon)(J_{Z_i} Y, J_{Z_i} Y) + \sum_{i=0}^{n-m-1} \mathrm{Hess}(r_\varepsilon)(W_i, W_i)$$

for appropriate bases $\{W_i\}$ of $\mathcal{H}$ and $\{Z_i\}$ of $\mathcal{V}$. This splitting corresponds again to the decomposition

$$\mathcal{H} = \mathrm{span}(Y) \oplus \mathcal{H}_{Sas} \oplus \mathcal{H}_{Riem}$$

Laplacian Comparisons

In each component of the horizontal decomposition we can use the previous comparisons on the Hessian to obtain

Theorem (Baudoin, Grong, M., & Rizzi '19)

Let $(\mathbb{M}, g, \mathcal{H})$ be an H-type foliation with parallel horizontal Clifford structure and satisfying the J^2 condition, and with nonnegative horizontal Bott curvature. Then there exists a $C > 4$ such that

$$\Delta_{\mathcal{H}} r_0 \leq \frac{n - m + 3 + C(m-1)}{r_0}$$

This is not sharp, but we can recover sharp estimates in each subspace.

Horizontal Holonomy (with F. Baudoin)

- We explore a notion of horizontal holonomy on H-type foliations, naturally associated to the Bott connection.
- On H-type submersions we can identify the horizontal holonomy with the Riemannian holonomy of the base space.
- From this one recovers a Berger-Simons-type classification.

Index theory for sub-Laplacian on H-type manifolds (with F. Baudoin and E. Grong)

- On H-type manifolds we can consider the Dirac operator

$$D_\varepsilon = d + \delta_\varepsilon$$

  associated to the Riemannian penalty metric $g_\varepsilon = g_\mathcal{H} \oplus \frac{1}{\varepsilon} g_\mathcal{V}$.
- By considering the limit $\varepsilon \to 0^+$, we can achieve a notion of index for the sub-Riemannian Laplacian

$$\Delta_\mathcal{H} = \lim_{\varepsilon \to 0^+} \Delta_\varepsilon = \lim_{\varepsilon \to 0^+} D_\varepsilon^2$$

Gauss-Bonnet theorem for surfaces in sub-Riemannian manifolds (with E. Grong)

- For a surface Σ embedded in a H-type manifold (or further generalization), we seek to define a notion of Gaussian and geodesic curvature in the $\varepsilon \to 0^+$ limit.
- One then formulates a sub-Riemannian Gauss-Bonnet theorem in terms of these quantities that recovers topological information about the surface.

New projects

- Connections on foliations
- Cheng-type rigidity theorem on Sasakian manifolds (with L. Rizzi)
- Observability and controlability of Schrodinger-type operators on H-type manifolds (with C. Fermanian-Kammerer)