H-type Foliations

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Differential Geometry

Recall that Differential geometry is the study of generalized Euclidean spaces. We generally have the following structures:

- Topological: Notion of "nearness" for points
- Manifold: Local homeomorphisms to $\mathbb{R}^n$
- Smooth: Derivatives of functions are well-defined

Riemannian Geometry

This is insufficient to allow for a notion of distance; the typical approach is to introduce a Riemannian metric.

Definition

To each point $p \in \mathbb{M}$ assign to the tangent space $T_p\mathbb{M}$ a symmetric positive-definite bilinear form g_p that smoothly varies with p.

We call g a Riemannian metric and the pair $(\mathbb{M}, g)$ a Riemannian manifold.

For a vector $v \in T_p\mathbb{M}$, we define a norm $\|v\|_p = \sqrt{g_p(v_p, v_p)}$.

Connections

Denoting the space of vector fields $\mathfrak{X}(\mathbb{M})$, an operator

$$\nabla \colon \mathfrak{X}(\mathbb{M}) \times \mathfrak{X}(\mathbb{M}) \to \mathfrak{X}(\mathbb{M})$$

such that

1. $\nabla_{fX+Y} Z = f \nabla_X Z + \nabla_Y Z$
2. $\nabla_X (fY) = df(X) Y + f \nabla_X Y$

is called a connection.

Curvature

Equipped with a connection, we can define curvature:

- Riemannian Curvature:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{\nabla_X Y - \nabla_Y X} Z$$

- Sectional Curvature: (for orthonormal X, Y):

$$S(X,Y) = g(R(X,Y)Y, X)$$

- Ricci Curvature:

$$Ric(X,Y) = Tr_g(Z \mapsto R(Z,X)Y)$$

- Scalar Curvature:

$$K = Tr_g(Ric)$$

Levi-Civita Connection

While there are many possible connections on a manifold, there is one in particular of interest in Riemannian geometry.

Theorem (Fundamental Theorem of Riemannian Geometry)

There exists a unique connection ∇ such that

1. $(\nabla_X g)(Y,Z) := X\cdot(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0$
2. $T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] = 0$

This connection is called the Levi-Civita connection.

Towards sub-Riemannian Motivations

If we regard curves as describing "motions" in a manifold, what happens when we impose restrictions? We can do this by requiring that curves only move in proscribed directions, given by a subbundle of the tangent space.

Definition

Let $\mathcal{H} \subseteq T\mathbb{M}$ be a constant rank subbundle. A curve $\gamma\colon [0,1] \to \mathbb{M}$ such that $\dot{\gamma} \in \mathcal{H}$ is called horizontal.

Suppose moreover that that sufficiently many brackets of vector fields in $\mathcal{H}$ generate the entire tangent space. We call E a bracket-generating distribution.

Theorem (Chow-Rashevskii)

Any two points in $\mathbb{M}$ can be connected by a horizontal curve if and only if $\mathcal{H}$ is bracket-generating.

Basic Definitions, sub-Riemannian Geometry

Let $\mathbb{M}$ be a smooth manifold. We say that

- $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$ is a sub-Riemannian manifold if
  - $\mathcal{H}$ is a constant rank, bracket generating subbundle of $T\mathbb{M}$,
  - and $g_{\mathcal{H}}$ is a inner product on $\mathcal{H}$.
- $(\mathbb{M}, \mathcal{H}, g)$ is a sub-Riemannian manifold with metric preserving complement or sRmc-manifold if
  - $(\mathbb{M}, g)$ is a Riemannian manifold,
  - the metric orthogonally splits as $g = g_{\mathcal{H}} \oplus g_{\mathcal{V}}$,
  - and $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$ is a sub-Riemannian manifold.

  We denote by $\mathcal{V}$ the orthogonal complement of $\mathcal{H}$ by g.

sub-Riemannian Geometry versus Riemannian Geometry

- Riemannian geometry is distinguished by the Riemannian metric g and thus a definition of distance in all directions.
- sub-Riemannian geometry has a notion of distance, but only in some directions.
- One major goal of sub-Riemannian geometry is to recover generalizations of Riemannian results.

Motivating Example: Hopf Fibration

Consider $\mathbb{S}^{2n+1}$ foliated as

$$\mathbb{S}^1 \xhookrightarrow{\iota} \mathbb{S}^{2n+1} \xrightarrow{\pi} \mathbb{C}P^n$$

We can understand horizontal distribution as

$$\mathcal{H}_x \cong T_{\pi(x)}\mathbb{C}P^n$$

the space $(\mathbb{S}^{2n+1}, \mathcal{H}, g)$ is an sRmc-manifold, foliated with leaves

$$\mathcal{V}_{\iota(x)} \cong T_x\mathbb{S}^1$$

Gromov-Hausdorff Convergence

For a sRmc-manfiold $(\mathbb{M}, \mathcal{H}, g)$ we define the canonical variation of the metric

$$g_\varepsilon = g_{\mathcal{H}} + \frac{1}{\varepsilon} g_{\mathcal{V}}$$

which in the Gromov-Hausdorff sense

$$(\mathbb{M}, \mathcal{H}, g_\varepsilon) \xrightarrow{\varepsilon \to 0^+} (\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$$

One key idea of this project is to leverage our knowledge of the Riemannian structure to understand the sub-Riemannian one.

Hladky-Bott Connection

Theorem (Hladky '12)

There exists a unique metric connection ∇ on $(\mathbb{M}, \mathcal{H}, g)$ such that

1. $\mathcal{H}$ *and* $\mathcal{V}$ *are* ∇*-parallel,*
2. *The torsion* T *of* ∇ *satisfies*
    - $T(\mathcal{H}, \mathcal{H}) \subset \mathcal{V}$,
    - $T(\mathcal{V}, \mathcal{V}) \subset \mathcal{H}$
3. *For every* $X, Y \in \Gamma(\mathcal{H}), Z, V \in \Gamma(\mathcal{V})$,
    - $\langle T(X,Z), Y\rangle_{\mathcal{H}} = \langle T(Y,Z), X\rangle_{\mathcal{H}}$
    - $\langle T(Z,X), V\rangle_{\mathcal{V}} = \langle T(V,X), Z\rangle_{\mathcal{V}}$.

This is called the Hladky-Bott connection.

Hladky-Bott Connection

We can explicitly write ∇ in terms of the Levi-Civita connection ∇^g as

$$\nabla_X Y = \begin{cases} \pi_{\mathcal{H}} \nabla^g_X Y & X, Y \in \Gamma(\mathcal{H}) \\ \pi_{\mathcal{H}}[X, Y] + A_X Y & Y \in \Gamma(\mathcal{H}), X \in \Gamma(\mathcal{V}) \\ \pi_{\mathcal{V}}[X, Y] + A_X Y & Y \in \Gamma(\mathcal{V}), X \in \Gamma(\mathcal{H}) \\ \pi_{\mathcal{V}} \nabla^g_X Y & X, Y \in \Gamma(\mathcal{V}) \end{cases}$$

where the tensor A is defined by

$$\langle A_X Y, Z \rangle = \frac{1}{2}\left((\mathcal{L}_{X_{\mathcal{V}}} g)(Y_{\mathcal{H}}, Z_{\mathcal{H}}) + (\mathcal{L}_{X_{\mathcal{H}}} g)(Y_{\mathcal{V}}, Z_{\mathcal{V}}) \right)$$

Bundle-like Metrics and Totally Geodesic Foliations

There are two important properties we will require:

- Bundle-like metric: A foliation is said to have a bundle-like metric if the metric locally splits orthogonally. This is equivalent to

$$\mathcal{L}_{\mathcal{V}} g(\mathcal{H}, \mathcal{H}) = 0$$

- Totally geodesic foliation: A foliation is said to be totally geodesic if the geodesics of the fibers are embedded geodesics of the total space. This is equivalent to

$$\mathcal{L}_{\mathcal{H}} g(\mathcal{V}, \mathcal{V}) = 0$$

J Map

On $(\mathbb{M}, \mathcal{H}, g)$ we can associate to each vector field $Z \in \Gamma(T\mathbb{M})$ an endomorphism J_Z of $T\mathbb{M}$ defined by

$$\langle J_Z X, Y \rangle = \langle Z, T(X, Y) \rangle$$

If $\mathcal{V}$ is integrable,

$$\begin{cases} J_Z X \in \mathcal{H} & \text{if } Z \in \mathcal{V}, X \in \mathcal{H} \\ J_Z X = 0 & \text{otherwise} \end{cases}$$

We thus take the perspective

$$J \colon \mathcal{V} \to \mathrm{End}(\mathcal{H}), \quad Z \mapsto J_Z$$

H-type Foliations

Definition

Let $(\mathbb{M}, \mathcal{H}, g)$ be a sRmc-manfiold. We say that $(\mathbb{M}, \mathcal{H}, g, J)$ is an H-type foliation if

1. $(\mathbb{M}, \mathcal{V}, g)$ is a totally geodesic foliation with bundle-like metric,
2. $\mathcal{V}$ is integrable, and
3. for all $X, Y \in \Gamma(\mathcal{H}), Z \in \Gamma(\mathcal{V})$,

$$\langle J_Z X, J_Z Y \rangle_{\mathcal{H}} = \|Z\|^2 \langle X, Y \rangle_{\mathcal{H}}$$

Parallel Torsion

We also refine the definition of H-type foliations based on the behavior of derivatives of the Hladky-Bott torsion T.

- If $\delta_{\mathcal{H}} T = 0$ we say $\mathbb{M}$ is of Yang-Mills type,
- If $\nabla_{\mathcal{H}} T = 0$ we say $\mathbb{M}$ has horizontally parallel torsion, and
- If $\nabla T = 0$ we say $\mathbb{M}$ has completely parallel torsion.

Lemma

All H-type foliations are Yang-Mills.

Example: Twistor Spaces

Let $(\mathbb{M}, g)$ be a $4n$-dimensional ($n \geq 2$) quaternionic-Kähler manifold, and fix a quaternionic structure E spanned by $\mathcal{I}, \mathcal{J}, \mathcal{K} \in \mathrm{End}(T\mathbb{M})$. Choosing a metric on E so that $\mathcal{I}, \mathcal{J}, \mathcal{K}$ are orthonormal, we define the twistor space over $\mathbb{M}$ to be the unit sphere bundle of E.

In this case, we have $\mathcal{H}_x \cong T_x\mathbb{M}$, $m = \dim \mathcal{V} = 2$, and there is a quaternionic structure induced by $\mathcal{V}$ acting on $\mathcal{H}$.

Examples

Structure	Torsion
Complex Type, $m = 1, n = 2k$	
K-Contact Manifolds	YM
Heisenberg Group, Hopf, Anti de-Sitter Fibrations	CP
Twistor Type, $m = 2, n = 4k$	
Twistor space over quaternionic Kähler manifold	HP
Projective Twistor space $\mathbb{CP}^1 \hookrightarrow \mathbb{C}P^{2k+1} \to \mathbb{H}P^k$	HP
Hyperbolic Twistor space $\mathbb{CP}^1 \hookrightarrow \mathbb{C}H^{2k+1} \to \mathbb{H}H^k$	HP
Quaternionic Type, $m = 3, n = 4k$	
3-Sasakian Manifolds	HP
Torus bundle over hyperkähler manifolds	CP
Quaternionic Heisenberg Group, Hopf, and Anti-de Sitter Fibrations	CP/HP
Octonionic Type, $m = 7, n = 8$	
Octonionic Heisenberg Group, Hopf, and Anti-de Sitter Fibrations	CP/HP
H-type Groups, m is arbitrary	CP

Dimensional Restrictions

For unit $Z \in \mathcal{V}$, the maps J_Z will induce complex, quaternionic, and octonionic structures on $\mathcal{H}$. As a consequence,

Lemma

Denote $m = \mathbf{rk}(\mathcal{V}), n = \mathbf{rk}(\mathcal{H})$. *Then*

1. $m \leq n - 1$,
2. $m = n - 1$ *implies* $n = 2, 4,$ *or* 8,
3. $n = 2k$, *and furthermore*
    - *if* $m \geq 2$ *then* $n = 4k$,
    - *if* $m \geq 4$ *then* $n = 8k$.

Clifford Structures

Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation with $Z_i, Z_j \in \mathcal{V}$, then

$$J_{Z_i}J_{Z_j} + J_{Z_j}J_{Z_i} = -2\langle Z_i, Z_j\rangle Id_{\mathcal{H}}$$

and so we can extend J in the natural way to

$$J\colon Cl(\mathcal{V}) \to \mathrm{End}(\mathcal{H})$$

There is a classification of such Clifford structures over Riemannian manifolds. (A. Moroianu, U. Semmelmann '11)

Parallel Horizontal Clifford Structures

Arising from Riemannian or semi-Riemannian foliations with curvature constancy we have the following notion:

Definition

Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation with horizontally parallel torsion. Then if there exists a map

$$\Psi \colon \mathcal{V} \times \mathcal{V} \to Cl_2(\mathcal{V})$$

such that

$$(\nabla_{Z_1} J)_{Z_2} = J_{\Psi(Z_1, Z_2)}$$

for all $Z_1, Z_2 \in \Gamma(\mathcal{V})$ then we say that $\mathbb{M}$ has a parallel horizontal Clifford structure.

This has a strong relationship with the horizontal holonomy of the space.

Parallel Horizontal Clifford Structures

Moreover, these structures are rigid in that they must be of the following form:

Lemma (Baudoin, Grong, Rizzi, & M. '18)

Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation with parallel horizontal Clifford structure. There exists $\kappa \in \mathbb{R}$ such that

$$\Psi(Z_1, Z_2) = -\kappa(Z_1 \cdot Z_2 + \langle Z_1, Z_2\rangle)$$

for all $Z_1, Z_2 \in \Gamma(\mathcal{V})$; moreover the sectional curvature of the leaves associated to $\mathcal{V}$ is constant and equal to κ^2.

The Einstein Property

Definition

Let $(\mathbb{M}, g)$ be a Riemannian manifold. We say that $\mathbb{M}$ is an Einstein manifold if there exists some constant $\lambda \in \mathbb{R}$ such that

$$\langle X, Y \rangle = \lambda \operatorname{Ric}(X, Y)$$

for all $X, Y \in \Gamma(T\mathbb{M})$.

Einstein Manifolds

- Originate in physics, as solutions to the Einstein field equations in vacuum.
- Mathematically, these are of great interest as model spaces.
- Ideal for computation, while still including a large class of structures.
- Include Euclidean space, complex projective spaces, and Calabi-Yau manifolds

Horizontal Ricci Curvature

Let $(\mathbb{M}, \mathcal{H}, g^\epsilon)$ be a sub-Riemannian manifold with metric preserving complement equipped with the canonical variation.

Unfortunately, in the limit $\epsilon \to 0^+$ the Ricci curvature associated to ∇ is not well defined. c.f. (Baudoin, Kim, Wang '16).

Horizontal Ricci Curvature

We define the horizontal and vertical Riemann curvature tensors by

$$R_{\mathcal{H}}(X, Y)Z = R^{\nabla}(X_{\mathcal{H}}, Y_{\mathcal{H}})Z_{\mathcal{H}},$$
$$R_{\mathcal{V}}(X, Y)Z = R^{\nabla}(X_{\mathcal{V}}, Y_{\mathcal{V}})Z_{\mathcal{V}},$$

from which it will follow that

$$R^{\nabla}(X, Y)Z = R_{\mathcal{H}}(X, Y)Z + R_{\mathcal{V}}(X, Y)Z + (\nabla_Z T)(X, Y)$$

Horizontal Ricci Curvature

Definition

The horizontal Ricci curvature $\mathrm{Ric}_{\mathcal{H}}$ is the horizontal trace of $R_{\mathcal{H}}$:

$$\mathrm{Ric}_{\mathcal{H}}(X, Y) = \sum_{i=1}^{n} R_{\mathcal{H}}(X, X_i, Y, X_i)$$

where $X_1, \cdots, X_n$ is a local orthonormal frame for $\mathcal{H}$.

The Horizontal Einstein Property

We can now define an analog of the Einstein property in our setting:

Definition

Let $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$ be a sub-Riemannian manifold. We say that $\mathbb{M}$ has the horizontal Einstein property if there exists some constant $\lambda \in \mathbb{R}$ such that

$$\langle X, Y \rangle_{\mathcal{H}} = \lambda \, \mathrm{Ric}_{\mathcal{H}}(X, Y)$$

for all $X, Y \in \Gamma(\mathcal{H})$.

The Horizontal Einstein Property

Theorem (Baudoin, Grong, Rizzi, & M. '18)

Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation with parallel horizontal Clifford structure, with $m \geq 2$. Then $\mathbb{M}$ is horizontally Einstein with

$$\lambda = \begin{cases} \kappa\left(\frac{n}{4} + 2(m-1)\right) & m \neq 3 \text{ or } m = 3 \text{ quaternionic} \\ \kappa\left(\frac{n}{4} + 4 + \frac{1}{4}D^{\mathcal{H}}\sigma\right) & m = 3 \text{ nonquaternionic} \end{cases}$$

where in the nonquaternionic case we have the orthogonal splitting $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, we define the constant $D^{\mathcal{H}} = \dim \mathcal{H}^+ - \dim \mathcal{H}^-$ and $\sigma = Id_{\mathcal{H}^+} \oplus (-Id_{\mathcal{H}^-})$.

J^2 condition

In many model cases, such as the Complex-, Quaternionic-, and Octonionic-Hopf fibrations we have a stronger property:

Definition

Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation. We say that it satisfies the J^2 condition if for every $Z_1, Z_2 \in \mathcal{V}$ with $\langle Z_1, Z_2 \rangle = 0$ there exists $Z_3 \in \mathcal{V}$ such that

$$J_{Z_1} J_{Z_2} = J_{Z_3}$$

The H-type groups with this property were classified by (M. Cowling, A.H. Dooley, A. Korányi, and F.Ricci '91)

Metric Connections, Jacobi Equation

For this section of the talk, let $(M, \mathcal{H}, g_\varepsilon)$

- be a sRmc-manifold,
- equipped with the canonical variation g_ε,
- having horizontally parallel torsion,
- and satisfying the J^2 condition.

For any $\varepsilon > 0$ the connection

$$\hat{\nabla}^\varepsilon_X Y = \nabla_X Y + J^\varepsilon_X Y$$

will be metric with metric adjoint ∇^ε. For a g_ε-geodesic γ, the Jacobi equation in this setting is

$$\hat{\nabla}^\varepsilon_{\dot{\gamma}} \nabla^\varepsilon_{\dot{\gamma}} W + \hat{R}^\varepsilon(W, \dot{\gamma})\dot{\gamma} = 0$$

The Comparison Principle

Theorem (Baudoin, Grong, Kuwada, & Thalmaier '17)

- *Let* $x, y \in \mathbb{M}$,
- $\gamma\colon [0, r_\varepsilon] \to \mathbb{M}$ *a unit speed* g_ε*-geodesic connecting* x, y, *and*
- $W_1, \cdots, W_k$ *be a collection of vector fields along* γ *such that*

$$\sum_{i=0}^{k} \int_0^{r_\varepsilon} \langle \hat{\nabla}^\varepsilon_{\dot\gamma} \nabla^\varepsilon_{\dot\gamma} W_i + \hat{R}^\varepsilon(W_i, \dot\gamma)\dot\gamma, W_i \rangle_\varepsilon \geq 0$$

then at $y = \gamma(r_\varepsilon)$ *it holds that*

$$\sum_{i=0}^{k} \mathrm{Hess}^{\hat{\nabla}^\varepsilon}(r_\varepsilon)(W_i, W_i) \leq \sum_{i=0}^{k} \langle W_i, \hat{\nabla}^\varepsilon_{\dot\gamma} W_i \rangle_\varepsilon$$

with equality if and only if the W_i *are Jacobi fields.*

Horizontal Splitting

We introduce an orthogonal splitting of the horizontal bundle. Fixing a vector field $Y \in \mathcal{H}$,

$$\mathcal{H} = \mathrm{span}(Y) \oplus \mathcal{H}_{Riem}(Y) \oplus \mathcal{H}_{Sas}(Y)$$

where

$$\mathcal{H}_{Sas}(Y) = \{J_Z Y | Z \in \mathcal{V}\}$$

$$\mathcal{H}_{Riem}(Y) = \{X \in \mathcal{H} | X \perp \mathcal{H}_{Sas} \oplus \mathrm{span}(Y)\}$$

Lemma

Denoting $n = \mathbf{rk}(\mathcal{H}), m = \mathbf{rk}(\mathcal{V})$, *we will have*

$$\dim(\mathcal{H}_{Sas}) = m, \quad \dim(\mathcal{H}_{Riem}) = n - m - 1$$

Riemannian Comparison Theorem

On a Riemannian manifold assume there exists κ such that

$$Ric(X,Y) \geq (n-1)\kappa g(X,Y)$$

- Laplacian (Rauch) Comparison Theorem:

$$\Delta r \leq \begin{cases} (n-1)\sqrt{\kappa}\cot(\sqrt{\kappa}r) & \kappa > 0 \\ \frac{n-1}{r} & \kappa = 0 \\ (n-1)\sqrt{|\kappa|}\coth(\sqrt{|\kappa|}r) & \kappa < 0 \end{cases}$$

- Bonnet-Meyers Diameter Estimates: If $\kappa > 0$ then

$$\mathrm{diam}(\mathbb{M}) \leq \frac{\pi}{\sqrt{\kappa}}$$

and the fundamental group of $\mathbb{M}$ must be finite.

Comparison Functions

Simliarly to the Riemannian case, we consider the comparision functions

$$F_{Riem}(r,\kappa) = \begin{cases} \sqrt{\kappa}\cot(\sqrt{\kappa}r) & \text{if } \kappa > 0 \\ \frac{1}{r} & \text{if } \kappa = 0 \\ \sqrt{|\kappa|}\coth(\sqrt{|\kappa|}r) & \text{if } \kappa < 0 \end{cases}$$

$$F_{Sas}(r,\kappa) = \begin{cases} \frac{\sqrt{\kappa}\left(\sin(\sqrt{\kappa}r) - \sqrt{\kappa}r\cos(\sqrt{\kappa}r)\right)}{2 - 2\cos(\sqrt{\kappa}r) - \sqrt{\kappa}r\sin(\sqrt{\kappa}r)} & \text{if } \kappa > 0 \\ \frac{4}{r} & \text{if } \kappa = 0 \\ \frac{\sqrt{\kappa}\left(\sqrt{\kappa}r\cosh(\sqrt{\kappa}r) - \sinh(\sqrt{\kappa}r)\right)}{2 - 2\cosh(\sqrt{\kappa}r) + \sqrt{\kappa}r\sinh(\sqrt{\kappa}r)} & \text{if } \kappa < 0 \end{cases}$$

These comparison functions will correspond to the splitting of $\mathcal{H}$.

Hessian Comparisons

Theorem (Baudoin, Grong, Rizzi, & M. '19)

- *Let* $\gamma : [0, r_\varepsilon] \to \mathbb{M}$ *be a* g_ε*-geodesic. Then*

$$\mathrm{Hess}(r_\varepsilon)(\dot\gamma, \dot\gamma) \leq \frac{\|\dot\gamma\|^2 \left(1 - \|\dot\gamma\|^2\right)}{r_\varepsilon}$$

- *If* $\mathrm{Sec}(X \wedge Y) \geq \rho > 0$ *for all unit* $X, Y \in \mathcal{H}_{Riem}(\dot\gamma)$, *then*

$$\mathrm{Hess}(r_\varepsilon)(X, X) \leq F_{Riem}(r_\varepsilon, K)$$

- *If* $\mathrm{Sec}(X \wedge J_Z X) \geq \rho > 0$ *for all unit* $X \in \mathcal{H}_{Sas}(\dot\gamma)$, *then*

$$\mathrm{Hess}(r_\varepsilon)(X, X) \leq F_{Sas}(r_\varepsilon, K)$$

Where K *is a constant depending on* $\rho, \varepsilon, \|\nabla_{\mathcal{V}} r_\varepsilon\|$, *and* $\|\nabla_{\mathcal{H}} r_\varepsilon\|$.

Horizontal Ricci Curvature

We can define the horizontal Ricci curvature as the trace of the Riemann tensor,

$$\begin{aligned} \mathrm{Ric}_{\mathcal{H}}(X,X) &= \sum_{i=0}^{n} \langle R^{\nabla}(W_i,X)X, W_i\rangle \\ &= \langle R^{\nabla}(Y,X)X, Y\rangle + \mathrm{Ric}_{Sas}(X,X) + \mathrm{Ric}_{Riem}(X,X) \end{aligned}$$

where the splitting corresponds to the decomposition

$$\mathcal{H} = \mathrm{span}(Y) \oplus \mathcal{H}_{Sas} \oplus \mathcal{H}_{Riem}$$

Diameter Estimates

Theorem (Baudoin, Grong, Rizzi, & M. '19)

Let $\rho > 0$. Then for unit $X \in \mathcal{H}$,

1. $$\frac{\mathrm{Ric}_{Riem}(X,X)}{n-m-1} \geq \rho \implies \mathrm{diam}_0(\mathbb{M}) \leq \frac{\pi}{\sqrt{\rho}}$$
2. $$\mathrm{Sec}(X \wedge J_Z X) \geq \rho \implies \mathrm{diam}_0(\mathbb{M}) \leq \frac{2\pi}{\sqrt{\rho}}$$
3. $$\frac{\mathrm{Ric}_{Sas}(X,X)}{m} \geq \rho \implies \mathrm{diam}_0(\mathbb{M}) \leq \frac{2\pi\sqrt{3}}{\sqrt{\rho}}$$

and in each case the fundamental group of $\mathbb{M}$ must be finite.

The first two of these are sharp, as they are achieved in the complex, quaternionic, and octonionic Hopf fibrations.

sub-Laplacian

Similarly to the horizontal Ricci curvature, we can define the sub-Laplacian as the trace of the Hessian. For the distance function r_ε along a geodesic γ with $Y = \nabla_{\mathcal{H}} r_\varepsilon$,

$$\Delta_{\mathcal{H}} r_\varepsilon = \sum_{i=0}^{n} \text{Hess}(r_\varepsilon)(W_i, W_i)$$

$$= \text{Hess}(r_\varepsilon)(Y, Y) + \sum_{i=0}^{m} \text{Hess}(r_\varepsilon)(J_{Z_i} Y, J_{Z_i} Y) + \sum_{i=0}^{n-m-1} \text{Hess}(r_\varepsilon)(W_i, W_i)$$

for appropriate bases $\{W_i\}$ of $\mathcal{H}$ and $\{Z_i\}$ of $\mathcal{V}$. This splitting corresponds again to the decomposition

$$\mathcal{H} = \text{span}(Y) \oplus \mathcal{H}_{Sas} \oplus \mathcal{H}_{Riem}$$

Laplacian Comparisons

In each component of the horizontal decomposition we can use the previous comparisons on the Hessian to obtain

Theorem (Baudoin, Grong, Rizzi, & M. '19)

Let $(\mathbb{M}, g, \mathcal{H})$ be an H-type foliation with parallel horizontal Clifford structure and satisfying the J^2 condition, and with nonnegative horizontal Bott curvature. Then there exists a $C > 4$ such that

$$\Delta_{\mathcal{H}} r_0 \leq \frac{n - m + 3 + C(m-1)}{r_0}$$

This is not sharp, but we can recover sharp estimates in each subspace.

Parallel Transport

Equipped with a connection, we can define the notion of parallel transport of vector fields.

Let γ be a curve in $\mathbb{M}$. We say that a vector field X is parallel along γ if

$$\nabla_{\gamma'} X = 0$$

Given a curve γ from p to q and a γ-parallel vector field X, we say that X_q is the parallel transport of X_p along γ, which we denote

$$\begin{aligned} \tau_\gamma \colon \ T_p\mathbb{M} &\to T_q\mathbb{M} \\ X_p &\mapsto \tau_\gamma X_p = X_q \end{aligned}$$

Holonomy

We can use this to construct a family of isomorphisms of the tangent spaces of interest:

Definition

Fix a point $p \in \mathbb{M}$ and consider all possible curves γ in $\mathbb{M}$ that both start and end at p. Associated to each γ there is an automorphism of $T_p\mathbb{M}$ determined by the parallel transport. We call the collection of automorphisms the Riemannian holonomy group of $\mathbb{M}$ at p, denoted $\mathbf{Hol}(\mathbb{M}, p)$.

Immediate Properties of the Holonomy Group

- We are justified in calling this a group, since concatenation of loops is again a loop.
- If γ is a path connecting p to q, we see that

$$\mathbf{Hol}(\mathbb{M}, p) = \tau_\gamma^{-1}\mathbf{Hol}(\mathbb{M}, q)\tau_\gamma$$

  and so the holonomy group $\mathbf{Hol}(\mathbb{M})$ is independent of the basepoint (on each path-connected component of $\mathbb{M}$).
- Since parallel transport is a linear isometry, it is clear that

$$\mathbf{Hol}(\mathbb{M}) \subseteq O(n)$$

Berger's Theorem

- deRham Theorem: We need only consider irreducible holonomy groups.
- Ambrose-Singer Theorem: We can reduce the study of holonomy groups to the study of curvature endomorphisms.
- Using Cartan's classification of irreducible linear real representations of real Lie groups, Berger classified all possible Riemannian manifolds by their holonomy:

Theorem (Berger)

Suppose $(\mathbb{M}, g)$ *is a Riemannian manifold such that* $\mathbf{Hol}^0(\mathbb{M})$ *is irreducible. Then at least one of the following is satisfied:*

- $\mathbf{Hol}^0(\mathbb{M})$ *acts transitively on the sphere, or*
- $\mathbb{M}$ *is a locally symmetric space of rank greater than or equal to 2.*

Symmetric Case

Definition

A Riemannian manifold $\mathbb{M}$ is called symmetric if for each point $p \in \mathbb{M}$ there exists an isometry $f_p \colon \mathbb{M} \to \mathbb{M}$ such that

- $f_p(p) = p$
- $T_p(f_p) = -Id_{T_p\mathbb{M}}$

There is a complete classification due to Cartan of all symmetric Riemannian spaces.

Symmetric Case

Theorem (Cartan)

For complete Riemannian manifolds $\mathbb{M}$, the following are equivalent:

- *The manifold is symmetric,*
- $\nabla R = 0$,
- *The manifold is a homogenous space $\mathbb{M} = G/H$ where G is a Lie group, and H is a compact subgroup of G.*

This relates to the holonomy group in the following way:

Theorem

For an irreducible simply connected symmetric manifold $\mathbb{M} = G/H$, we have that $\mathbf{Hol}(\mathbb{M}) = H$ acting by the adjoint representation.

Nonsymmetric Case

Theorem (Berger, Simons)

For an irreducible simply connected nonsymmetric manifold $\mathbb{M}$, one of the following cases occurs:

$dim(\mathbb{M})$	$\mathbf{Hol}^0(\mathbb{M})$	*Type*
n	$O(n)$	*Generic*
n	$SO(n)$	*Oriented*
$n = 2m$	$U(m)$	*Kähler*
$n = 2m$	$SU(m)$	*Calabi-Yau*
$n = 4m$	$Sp(m) \cdot Sp(1)$	*Quaternion-Kähler*
$n = 4m$	$Sp(m)$	*Hyperkähler*
$n = 7$	G_2	
$n = 8$	$Spin(7)$	

H-type Submersions

In the case that the H-type foliation $(\mathbb{M}, \mathcal{H}, g)$ is induced by a global submersion

$$F \hookrightarrow \mathbb{M} \to \mathbb{B}$$

we refer to the structure $(\mathbb{M}, \mathcal{H}, g, \mathbb{B})$ as an H-type submersion with fiber F.

- These are models for the H-type foliations in the sense that every foliation is locally a submersion.
- We have a complete classification of H-type submersions with horizontally parallel Clifford structure.

Classification of H-type submersions with horizontally parallel Clifford structure, $\kappa > 0$

$\mathbb{M}$	$\mathbb{B}$	Fiber	**rank**$(\mathcal{H})$	**rank**$(\mathcal{V})$
Twistor space	Quaternion-Kähler with positive scalar curvature	$\mathbb{S}^2$	$4k$	2
3-Sasakian	Quaternion-Kähler with positive scalar curvature	$\mathbb{S}^3$	$4k$	3
Quaternion-Sasakian	Product of two quaternion-Kähler with positive scalar curvature	$\mathbb{R}P^3$	$4k$	3
$\frac{\mathbf{Sp}(q^+ +1)\times\mathbf{Sp}(q^- +1)}{\mathbf{Sp}(q^+)\times\mathbf{Sp}(q^-)\times\mathbf{Sp}(1)}$	$\mathbb{H}P^{q^+}\times\mathbb{H}P^{q^-}$	$\mathbb{S}^3$	$4(q^+ + q^-)$	3
$\frac{\mathbf{Sp}(k+2)}{\mathbf{Sp}(k)\times\mathbf{Spin}(4)}$	$\frac{\mathbf{Sp}(k+2)}{\mathbf{Sp}(k)\times\mathbf{Sp}(2)}$	$\mathbb{S}^4$	$8k$	4
$\frac{\mathbf{SU}(k+4)}{\mathbf{S}(\mathbf{U}(k)\times\mathbf{Sp}(2)\mathbf{U}(1))}$	$\frac{\mathbf{SU}(k+4)}{\mathbf{S}(\mathbf{U}(k)\times\mathbf{U}(4))}$	$\mathbb{R}P^5$	$8k$	5
$\frac{\mathbf{SO}(k+8)}{\mathbf{SO}(k)\times\mathbf{Spin}(7)}$	$\frac{\mathbf{SO}(k+8)}{\mathbf{SO}(k)\times\mathbf{SO}(8)}$	$\mathbb{R}P^7$	$8k$, $k\geq 3$, k odd	7
$\frac{\mathbf{Spin}(k+8)}{\mathbf{SO}(k)\times\mathbf{Spin}(7)}$	$\frac{\mathbf{SO}(k+8)}{\mathbf{SO}(k)\times\mathbf{SO}(8)}$	$\mathbb{S}^7$	$8k$, $k=1$, k even	7
Exceptional cases				
$\frac{F_4}{\mathbf{Spin}(8)}$	$\frac{F_4}{\mathbf{Spin}(9)}=\mathbb{O}P^2$	$\mathbb{S}^8$	16	8
$\frac{E_6}{\mathbf{Spin}(8)\mathbf{U}(1)}$	$\frac{E_6}{\mathbf{Spin}(10)\mathbf{U}(1)}=(\mathbb{C}\otimes\mathbb{O})P^2$	$\mathbb{S}^9$	32	9
$\frac{E_7}{\mathbf{Spin}(11)\mathbf{SU}(2)}$	$\frac{E_7}{\mathbf{Spin}(12)\mathbf{SU}(2)}=(\mathbb{H}\otimes\mathbb{O})P^2$	$\mathbb{S}^{11}$	64	11
$\frac{E_8}{\mathbf{Spin}(15)}$	$\frac{E_8}{\mathbf{Spin}^+(16)}=(\mathbb{O}\otimes\mathbb{O})P^2$	$\mathbb{S}^{15}$	128	15

Horizontal Holonomy

Because the Bott connection preserves the horizontal distribution (independently of ε) we can understand that H-type foliations have a notion of horizontal holonomy.

Definition

Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation. Fix a point $p \in \mathbb{M}$, and consider all loops γ starting and ending at p. Each curve defines an isometry

$$\tau_\gamma : \mathcal{H}_p \to \mathcal{H}_p$$

the collection of which forms a group under concatenation, called the horizontal holonomy group $\mathbf{Hol}^\nabla(\mathbb{M}, \mathcal{H})$ of $(\mathbb{M}, \mathcal{H}, g)$.

This is independent of the choice of $p \in \mathbb{M}$.

Horizontal Holonomy of H-type Submersions

In particular, the holonomy of H-type submersions can be understood very concretely:

Theorem

Let $(\mathbb{M}, \mathcal{H}, g, \mathbb{B})$ be an H-type submersion. Then

$$\mathbf{Hol}^{\nabla}(\mathbb{M}, \mathcal{H}) \cong \mathbf{Hol}(\mathbb{B})$$

where $\mathbf{Hol}(\mathbb{B})$ *denotes the Riemannian holonomy group of* $\mathbb{B}$ *equipped with the Levi-Civita connection induced by* $g_{\mathcal{H}}$.

List of Horizontal Holonomy Groups

From this, we are able to ascertain a complete list of horizontal holonomy groups that occur for H-type submersions; For those with $\kappa > 0$:

$$\mathbb{S}(n)\mathbb{S}(1)$$
$$\mathbb{S}(n) \times \mathbb{S}(2)$$
$$S(U(n) \times U(4))$$
$$SO(n) \times SO(8)$$
$$Spin(9)$$
$$Spin(10)U(1)$$
$$Spin(12)SU(2)$$
$$Spin^{+}(16)$$

Thank you for your attention!